One-Way Inversibility of Functional Operators with a Shift in the Spaces L_P (Γ)

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ABSTRACT

Criteria are obtained for one-sided invertibility of functional operators with forward shift in Lebesgue spaces when an arbitrary non-empty set of periodic points has a shift.

KEYWORDS: *shift, one-sided shift, reversibility, one-way reversibility, periodic points*

Let – Γ be a simple closed smooth oriented curve of the complex plane, $\alpha(t)$ – be a diffeomorphism (shift) of the contour Γ onto itself, preserving the orientation (line) and having a non-empty set Λ of periodic points of multiplicity m.

In this paper, in the spaces $L_p(\Gamma)$, $1 \leq p < \infty$, we study a functional operator with a shift

$$A = a(t)I - b(t)W$$

where $a(t), b(t) \in C(\Gamma), I$ – single operator, W – shift operator: $(W_\varphi(t) = \varphi[\alpha(t)], t \in \Gamma$.

To date, within the framework of several different approaches, results have been obtained on the determination of the criterion of reversibility and one-sided reversibility of the operator A $(see. [1-7])$under various assumptions regarding the carrier contour and shear.

In [8], obtain a criterion for the one-sided invertibility of an operator A with a shift with two break points (p. 214) satisfying some conditions. In [9], the operator A was studied in the case of a code, the shift is a diffeomorphism and has a finite number of periodic points.

In this paper, we obtain criteria for the one-sided invertibility of the operator A in the space $L_p(\Gamma)$, $1 < p < \infty$ in the case of a code, the shift α has an arbitrary non-empty set of periodic points.

As is known (see, for example, [10]. P. 24-29), all periodic shift points α have the same multiplicity (period) m.

By $\Phi = Supp[\tau - \alpha_m(\tau)]$ we denote the closure of the set of all points of Γ at which $\alpha_m(\tau) \neq \tau$. For $u(t), a(t), b(t) \in C(\Gamma)$, we introduce the notation:

$$u_\pm(t) = \lim_{n \to \pm\infty} \prod_{i=0}^{m-1} u[\alpha_{n+i}(t)], \ \ h_\pm(t) = |a_\pm(t)| - |\alpha'_\pm(t)|^{-\frac{1}{p}} |b_\pm(t)|,$$

$$\Gamma_1 = \Gamma \backslash \Phi, \ \ \Gamma_2 = \{t \in \Phi: h_\pm(t) > 0\}$$

$\Gamma_3 = \{t \in \Phi: h_\pm(t) < 0\}, \ \ \Gamma_4 = \{ t \in \Phi: h_+(t) < 0 < h_-(t),$

$$\Gamma_5 = \{t \in \Phi: h_+(t) > 0 > h_-(t)\}$$

$$\nu_A(t)=\begin{cases}\prod_{i=0}^{m}a(\alpha_i(t))-\prod_{i=0}^{m-1}b(\alpha_i(t)),t\in\Gamma_1\\ \prod_{i=0}^{m-1}a(\alpha_i(t)),t\in\Gamma_2\\ \prod_{i=0}^{m-1}b(\alpha_i(t)),t\in\Gamma_3\\ 0,t\in\Gamma\backslash\bigcup_{i=1}^{3}\Gamma_i\end{cases}$$

It is easy to see that at periodic points t the equalities hold

$$h_{\pm}(t)=h_m(t)\stackrel{\text{def}}{=}|a_m(t)|-\left|\alpha_m'(t)\right|^{-\frac{1}{p}}\cdot|b_m(t)|$$

In the general case, Γ_1 by definition are an open set of points of the contour Γ at which $\alpha_m(t)=t$.

As is known, such a set Γ_1 can be represented as the sum of an at most countable set of open arcs $\hat{\gamma}$ on which the restriction of the shift $\alpha_m(t)$ is Carleman. Then Γ_1 can be represented in the form of unions of an at most countable collection of the set

$$\widehat{\Gamma}_i=\sum_{k=0}^{m-1}\alpha_k(\hat{\gamma}_i)$$

The set $\mathcal{T}=\Phi\backslash\Lambda$ is an open set representable as a sum of at most countable set of open arcs $\tilde{\gamma}$. There are no periodic points inside the arcs $\tilde{\gamma}$, and their ends τ_- and τ_+ are periodic points of the translation α. Then $\mathcal{T}$ is also represented as a union of an at most countable collection of sets

$$\widetilde{\Gamma}_i=\sum_{k=0}^{m-1}\alpha_k(\tilde{\gamma}_i).$$

If α has a finite number of periodic points, then Theorem -1 in [9] can be reformulated as follows:

Theorem -1. The operator A is invertible from the right (left) if and only if the conditions

$$\nu_{\mathrm{A}}(t)\neq 0,\qquad \forall t\in\Gamma\backslash\Gamma_4\ (\forall t\in\Gamma\backslash\Gamma_5)$$

and the set $\Gamma_4(\Gamma_5)$ satisfies the conditions

$\forall t\in\Gamma,\exists k_0\in\mathbb{Z},\ b(\alpha_k(t))\neq 0\ at\ k\geq k_0,\ a(\alpha_k(t))\neq 0\ at\ k<0,$

(respectively

$$\forall t\in\Gamma_5,\exists k_0\in\mathbb{Z},b(\alpha_k(t))\neq 0\ at\ k<k_0,\qquad a(\alpha_k(t))\neq 0\ \ at\ k>k_0)$$

from this theorem, using the methods of [9], one can prove the following assertion.

Lemma 1. Let α have a finite number of arcs of type $\hat{\gamma}$ and a finite number of periodic points belonging to the set $\Gamma\backslash\Gamma_1$. In this case, the operator A is right (left) invertible if and only if the conditions of Theorem 1 are satisfied.

Now let $\alpha(t)$ have a finite or countable number of arcs of type $\hat{\gamma}$ and $\hat{\gamma}$, $N_0'-$ is the derived set for $N_0=\Phi\cap\Lambda$.

Lemma 2. If A is right (left) invertible in the space $L_p(\Gamma)$, then the conditions

$h_m(\tau) \neq 0,\ \forall\tau \in N_o$ *(1)*

We carry out the proof for the case of right invertibility of the operator A (left invertibility is considered similarly).

Let $\tau \in N_0 \backslash N_0^{'}$. Then τ is the endpoint of the arc $\tilde{\gamma}$ that is invariant with respect to the translation α_m. Restricting A to the-invariant space

$$L_p(\bigcup_{k=0}^{m-1} \alpha_k(\tilde{\gamma}),$$

we obtain that A is invertible in this space from the right, but then, according to Theorem -1, $h_m(\tau) \neq 0$.

Suppose now that $\exists\tau_0 \in N_0^{'},\ h_m(\tau_0) = 0$. Then

$$h_m\big(\alpha_i(\tau_0)\big) = 0, \qquad i = 1,2, \dots ,m-1. \qquad (2)$$

For arbitrary $\varepsilon > 0$, there exist neighborhoods δ_i of points $\alpha_i(\tau_0), i = \overline{0, m-1}$ such that the sum of the length of all intervals δ_i does not exceed ε. Since $\tau_0 \in N_0^{'}$ is a point of condensation of arcs of type $\tilde{\gamma}$, then inside the arcs δ_0 one can choose two periodic points $\tau_0^{'}, \tau_0^{''}$ of the shift $\alpha(t)$ such that inside the arcs $(\tau_0^{'}, \tau_0^{''})$ there were no other periodic points of shift α and the point $\tau_0^{'}$ preceded the point $\tau_0^{''}$ in the direction of the contour Γ.

For definiteness, suppose that

$$\lim_{n\to+\infty} \alpha_{mn}(t) = \tau_0^{''}$$

for some point $t \in \left(\tau_0^{'}, \tau_0^{''}\right)$. Then the set

$$\theta = \sum_{i=0}^{m-1} (\alpha_i(\tau_0^{'}) \ \ \alpha_i(\tau_0^{''}))$$

belongs to the set

$$H = \bigcup_{i=0}^{m-1} \delta_i.$$

It is known ([10]. Pp. 23-28) that

$$\lim_{n\to\pm\infty} \alpha_{mn}(x)$$

exists for all $x \in \left(\alpha_i(\tau_0^{'}), \alpha_i(\tau_0^{''})\right)$, does not depend on the choice of points x and tends to $(\alpha_i(\tau_0^{'})$ as $n \to -\infty$ and in $\alpha_i(\tau_0^{''})$ as $n \to +\infty$, $i = 0,1,2, \dots, m-1$.

Since the operator A is invertible in $L_p(\Gamma)$ and the set θ is invariant with respect to α, it is also invertible on the right in $L_p(\theta)$, which are restrictions of the space $L_p(\Gamma)$ to the set θ. Therefore, by virtue of (2) and the stability of the one-sided invertibility property of operators, choosing ε small enough, we can perturb the coefficients of the operator A so that the condition

$$h_m(\tau_0^{'}) < 0 < h_m(\tau_0^{''}) \qquad (3)$$

and the perturbed operator A$'$ remains also invertible on the right in the space $L_p(\theta)$.

Since $A^{'}$ is right invertible in the space $L_p(\theta)$, for this the operators must be satisfied in the space $L_p(\theta)$ of the conditions of Theorem 1 in [9]. But conditions (3) contradict the conditions of Theorem

1 in [9]. Lemma is proven.

Theorem -2. The operator A is invertible from the right (left) to $L_p(\Gamma)$, $1 < p < \infty$, if and only if

$\nu_A(t) \neq 0, \forall t \in \Gamma \backslash \Gamma_4 \ (\forall t \in \Gamma \backslash \Gamma_5)$ (4)

and on the set $\Gamma_4(\Gamma_5)$ the condition

$\forall t \in \Gamma_4, \exists k_0 \in \mathbb{Z},\ b(\alpha_k(t)) = 0 \ npu \ k \geq k_0,\ a(\alpha_k(t)) \neq 0 \ npu \ k > k_0$ (5)

(respectively

$\forall t \in \Gamma_5, \exists k_0 \in \mathbb{Z}, b\ (\alpha_k(t)) \neq 0 \ npu \ k < k_0, a(\alpha_k(t)) \neq 0 \ npu \ k > k_0)$ (6)

We carry out the proof for the case of right invertibility of the operator (the case of left invertibility is considered similarly).

Need. If A is right invertible, then, according to Lemma 1, inequality (1) holds.

And then, by virtue of the definition of the sets $\Gamma_i, i = \overline{1,5}$, the contour Γ satisfies the equalities

$$\Gamma = \bigcup_{i=1}^{5} \Gamma_i$$

The set Γ_1 is represented as a union of at most countable collection of sets

$$\widehat{\Gamma}_j = \sum_{k=0}^{m-1} \alpha_k(\hat{\gamma}_j)$$

on which the restriction of the shift $\alpha(t)$ is Carleman. Similarly, the set $T = \Phi \backslash \Lambda$ is also represented as a union of an at most countable collection of $\alpha -$ invariant sets

$$\widetilde{\Gamma}_j = \sum_{k=0}^{m-1} \alpha_k\ (\tilde{\gamma}_j)$$

on which the restriction of the shift α is non-Carleman. Then

$$\Gamma = \bigcup_{i=1}^{5} \Gamma_i = (\bigcup_j \widetilde{\Gamma}_j) \cup (\bigcup_j \widehat{\Gamma}_j) \cup N_0$$

If the operator A is right invertible in the space $L_p(\Gamma)$, then it is also right invertible in the spaces $L_p(\widetilde{\Gamma}_j), L_p(\widehat{\Gamma}_j)$ and $L_p(N_0)$. Hence, according to Theorem -1, conditions (4) - (5) are satisfied for the sets $\widetilde{\Gamma}_j$ with Γ replaced by $\overline{\widetilde{\Gamma}_j}$ *and* Γ_4 by $\Gamma_4 \cap \widetilde{\Gamma}_j$. On the sets $\widehat{\Gamma}_i$ and $N_0 \subset \Gamma_2 \cup \Gamma_3$ (here we take into account that $h_{\pm}(\tau) = h_m(\tau)$ and $h_m(\tau) \neq 0$ according to (1)) the shift α is Carleman and again conditions (4) - (5) The necessity is proved.

Adequacy. Let the conditions of the theorem be satisfied. Then, at the ends τ of arcs from $\widehat{\Gamma}_j$ $\alpha_m'(\tau) = 1$, and hence $|a_m(\tau)| \neq |b_m(\tau)|$. Taking this into account, under the conditions of the theorem, the operator A is invertible on the right in each space $L_p(\widetilde{\Gamma}_j)$ *and* $L_p(\widehat{\Gamma}_j)$.

Take an arbitrary point z of the set N_0. In it, the values $h_m(z)$ do not vanish. Indeed, since $h_{\pm}(z) = h_m(z)$, then either $z \in \Gamma_2 \cup \Gamma_3$ or

$$z \in \Gamma \backslash \bigcup_{i=1}^{5} \Gamma_i$$

In the latter case, $\nu_\alpha(a,b) = 0$, which contradicts the hypothesis of the theorem. Therefore, $z \in \Gamma_2 \cup \Gamma_3$ and therefore $h_m(z) \neq 0$.

Further, by virtue of the continuity of the coefficients of the operator A at all points belonging to a sufficiently small neighborhood U_z of the point $z \in N_0' \subset N_0, h_+(t)$ *and* $h_-(t)$ have the same sign, i.e. or

$h_\pm(t) > 0,\ \forall t \in U_z$ (7)

or

$h_\pm(t) < 0,\ \forall t \in U_z$ (8)

Due to the compactness of the set N_0' and according to (7), (8), from any infinite covering of the set N_0', one can choose a finite subcover $U_i, i = \overline{1,\mu}$ such that at all periodic points $\tau \in U$_i either

$|a_m(\tau)| > |b_m(\tau)|\left|\alpha_m'(\tau)\right|^{-\frac{1}{p}}$ (9)

or

$|a_m(\tau)| < |b_m(\tau)| \cdot \left|\alpha_m'(\tau)\right|^{-\frac{1}{p}}$ (10)

Moreover, at all periodic points $\tau \in \Gamma_1 \cap U_i$, due to the equality $\alpha_m'(\tau) = 1$, either

$|a_m(\tau)| > |b_m(\tau)|$ (11)

or

$|a_m(\tau)| < |b_m(\tau)|$ (12)

Without loss of generality, we can assume that the boundary of the sets U_i consists of periodic shift points α. Then the sets

$$\beta_i = \bigcup_{k=0}^{m-1} \alpha_k(U_i), i = \overline{1,k_0}, k_0 = \frac{\mu}{m} \quad (13)$$

are invariant with respect to the shift α. It is easy to see that (7) or (8) also hold for any point t belonging to the set $\beta_i \cap \Gamma$. Suppose that (7) holds (if (8) holds, then the reasoning is similar). Then

$$\prod_{j=0}^{m-1} a\left(\alpha_j(t)\right) \neq 0,\ \forall t \in \beta_i, \qquad i = \overline{1,k_0}\ \ (k_0 = \frac{\mu}{m})$$

and since the coefficients of the operator A are continuous, for any $\varepsilon > 0$ from the covering of the set N_0' one can choose a finite covering U_i, such that

$$\left|\frac{b_m(t)}{a_m(t)}\right| \leq \left|\frac{b_m(\tau)}{a_m(\tau)}\right| + \varepsilon, \qquad \left|\alpha_m'(t)\right|^{-\frac{1}{p}} \leq (1+\varepsilon)^{-\frac{1}{p}}, \forall t \in U_i \quad (14)$$

where τ is one of the ends of one of the arcs U_i.

Let us estimate the spectral radius $\rho_i(T_g)$ of the operator $T_g = gW$, where

$g(t) = \frac{b(t)}{a(t)}$ in the space $L_p(\beta_i)$.

Taking into account (14), for $n = km$,

$$\left\|\left(T_g^n\varphi\right)(t)\right\|_{L_p(\beta_i)} \leq (|g_m(\tau)| + \varepsilon)^k \cdot (1+\varepsilon)^{-\frac{1}{p}})\|\varphi\|_{L_p(\beta_i)}$$

Then, in the space $L_p(\beta_i)$,

$$\left\|T_g^n\right\| \leq (|g_m(\tau)| + \varepsilon)^{\frac{n}{m}} (1+\varepsilon)^{-\frac{1}{p}} \qquad (15)$$

U_3 (15) it follows that $\rho_i(T_g) \leq (|g_m(\tau)| + \varepsilon)^{\frac{1}{m}}$ in the space $L_p(\beta_i)$. Taking into account (7) and choosing ε small enough, we achieve the inequality $\rho_i(T_g) < 1$. Then, as is easy to see, the spectral radius $\rho(T_g)$ of the operators T_g in the space $L_p(\theta)$, where

$$\theta = \bigcup_{i=1}^{k_0} \beta_i$$

also less than one. Then, according to the well-known theorem on the inverse operator, the operator A is invertible in the space $L_p(\theta)$.

The space $L_p(\Gamma)$ is decomposed by the direct sum of the subspaces $L_p(\theta)$ and $L_p(\Gamma\backslash\theta)$, that are invariant with respect to α. In the space $L_p(\Gamma\backslash\theta)$, the operator A has a finite number of periodic points and a finite number of arcs of type $\hat{\gamma}$. Therefore, according to Lemma 1, under the condition of the theorem, the operator A is invertible on the right in $L_p(\Gamma\backslash\theta)$. Therefore, A is invertible on the right in $L_p(\Gamma\backslash\theta)$. The theorem is completely proved.

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