Optimal Quadrature Formulas with Derivatives in the Space $W_2^{(6,5)}$

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ABSTRACT
In the paper we consider an extension problem of the Euler-Maclaurin quadrature formula in the space by constructing an optimal quadrature formula.

KEYWORDS: *Optimal quadrature formula, Hilbert space, the error functional, Sobolev method, discrete argument function.*

1. Statement of the problem.

Consider the following quadrature formula

$$\int_0^1 \varphi(x)dx \cong \sum_{\beta=0}^{N}\left(C_0[\beta]\varphi(h\beta)+C_1[\beta]\varphi'(h\beta)+C_2[\beta]\varphi''(h\beta)+C_5[\beta]\varphi^{(5)}(h\beta)\right) \quad (1)$$

where

$$C_0[0]=\frac{h}{2},\quad C_0[\beta]=h,\quad C_0[N]=\frac{h}{2},\quad (\beta=1,2,...,N-1) \quad (2)$$

$$C_1[0]=\frac{h^2}{12},\quad C_1[\beta]=0,\quad C_1[N]=-\frac{h^2}{12},\ (\beta=1,2,...,N-1), \quad (3)$$

and

$$C_2[0]=-\frac{h^4}{720},\quad C_2[\beta]=0,\quad C_2[N]=\frac{h^4}{720},\quad (\beta=1,2,...,N-1) \quad (4)$$

$C_5\left[\beta\right]$ are unknown coefficients of the quadrature formula (1), $h=\frac{1}{N}$, N is a natural number.

We suppose that integrands φ belong to $W_2^{(6,5)}$, where by $W_2^{(6,5)}$ is the class of all functions φ defined on [0; 1] which posses an absolutely continuous sixth derivative and whose fifth derivative is in $L_2(0,1)$ (see [4]). The class $W_2^{(6,5)}$ under the pseudo-inner product

$$\langle\varphi,\psi\rangle_{W_2^{(6,5)}}=\int_0^1(\varphi^{(6)}(x)+\varphi^{(5)}(x))(\psi^{(6)}(x)+\psi^{(5)}(x))dx$$

is a Hilbert if we identify functions that differ by a linear combination of $1,\ x,\ x^2,\ x^3,\ x^4$ and e^{-x}

(see, for example, [1, 4]). The space $W_2^{(6,5)}$ is equipped by the corresponding norm

$$\|\varphi\|_{W_2^{(6,5)}} = \left[\int_0^1 (\varphi^{(6)}(x) + \varphi^{(5)}(x))^2 dx\right]^{1/2}. \quad (5)$$

The error of the formula (1) is the difference

$$(\ell,\varphi) = \int_0^1 \varphi(x)dx - \sum_{\beta=0}^{N}\left(C_0[\beta]\varphi(h\beta) + C_1[\beta]\varphi'(h\beta) + C_2[\beta]\varphi'''(h\beta) + C_5[\beta]\varphi^{(4)}(h\beta)\right) \quad (6)$$

and it defines a functional

$$\ell(x) = \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} C_0[\beta]\delta(x-h\beta) - C_1[\beta]\delta'(x-h\beta) - \\ -C_2[\beta]\delta'''(x-h\beta) - C_5[\beta]\delta^{(4)}(x-h\beta) \quad (7)$$

which is called *the error functional* of the quadrature formula (1), where $\varepsilon_{[0,1]}(x)$ is the indicator of the interval [0; 1], δ is Dirac's delta-function. The value of the error functional ℓ at a function φ is calculated as $(\ell,\varphi) = \int_{-\infty}^{\infty} \ell(x)\varphi(x)dx$ (see [7]) and ℓ is a linear functional in $W_2^{(6,5)*}$ space, where $W_2^{(6,5)*}$ is the conjugate space to the space $W_2^{(6,5)}$.

Since the error functional (7) is defined on the space $W_2^{(6,5)}$ it is necessary to impose the following conditions for the error functional ℓ

$$(\ell,1) := 1 - \sum_{\beta=0}^{N} C_0[\beta] = 0, \quad (8)$$

$$(\ell,x) := \frac{1}{2} - \sum_{\beta=0}^{N}\left(C_0[\beta](h\beta) + C_1[\beta]\right) = 0, \quad (9)$$

$$(\ell,x^2) := \frac{1}{3} - \sum_{\beta=0}^{N}\left(C_0[\beta](h\beta)^2 + 2C_1[\beta](h\beta)\right) = 0, \quad (10)$$

$$(\ell,x^3) := \frac{1}{4} - \sum_{\beta=0}^{N}\left(C_0[\beta](h\beta)^3 + 3C_1[\beta](h\beta)^2 + 6C_2[\beta]\right) = 0, \quad (11)$$

$$(\ell,x^4) := \frac{1}{5} - \sum_{\beta=0}^{N}\left(C_0[\beta](h\beta)^4 + 4C_1[\beta](h\beta)^3 + 24C_2[\beta](h\beta)\right) = 0, \quad (12)$$

$$(\ell,e^{-x}) := \int_0^1 e^{-x}dx - \sum_{\beta=0}^{N}\left(C_0[\beta]e^{-h\beta} - C_1[\beta]e^{-h\beta} - C_2[\beta]e^{-h\beta} - C_5[\beta]e^{-h\beta}\right) = 0. \quad (13)$$

The equations (8)-(13) show that the quadrature formula (1) is exact to the functions $1, x, x^2, x^3$ and e^{-x}. One can see that the coefficients $C_0[\beta]$, $C_1[\beta]$, $C_2[\beta]$ and $C_5[\beta]$ defined by equalities (2), (3) and (4) satisfy equations (8), (9), (10), (11) and (12). Therefore for unknown coefficients $C_5[\beta], \beta = 0,1,...,N$, we have only equation (13).

The absolute value of the error (6) is estimated from above by the norm

$$\|\ell\|_{W_2^{(6,5)*}} = \sup_{\varphi, \|\varphi\|_{W_2^{(6,5)}} \neq 0} \frac{|(\ell,\varphi)|}{\|\varphi\|_{W_2^{(6,5)}}}$$

of the error functional ℓ as follows

$$|(\ell,\varphi)| \le \|\varphi\|_{W_2^{(6,5)}} \cdot \|\ell\|_{W_2^{(6,5)*}}.$$

Furthermore, one can see from (6) that the norm of the error functional (7) depends on coefficients $C_5[\beta]$.

Thus, in order to construct *an optimal quadrature formula* of the form (1) in the sense of Sard we have to solve the following problem.

Problem 1. Find the minimum for the norm of the error functional (7) by coefficients $C_5[\beta]$, i.e.

$$\|\ell\|_{W_2^{(6,5)*}} = \inf_{C_4[\beta]} \|\ell\|_{W_2^{(5,4)*}}. \qquad (14)$$

The coefficients satisfying equality (14) are called *optimal coefficients* and are denoted as $C_5[\beta]$, $\beta = 0,1,...,N$.

For solving Problem 1, first, we find an expression for the norm of the error functional (7) and next, we calculate it's minimum by coefficients $C_5[\beta]$, $\beta = 0,1,...,N$.

The rest of the paper is organized as follows:

2. Coefficients of the optimal quadrature formula.

2.1.The norm of the error functional ℓ.

To get a representation of the norm of the error functional (7) in the space $W_2^{(6,5)}(0,1)$ we use *the extremal function* for this functional (7) which satisfies the following equality (see [7, 8]):

$$(\ell,\psi_\ell) = \|\ell\|_{W_2^{(5,4)*}} \cdot \|\psi_\ell\|_{W_2^{(5,4)}}.$$

From Theorem 2.1 of the work [4] for the extremal function ψ_ℓ of the error functional ℓ in the space $W_2^{(6,5)}$ we get the following formula

$$\psi_\ell(x) = \ell(x) * G_6(x) + P_4(x) + de^{-x}, \qquad (15)$$

where

$$G_6(x) = \frac{\mathrm{sgn}x}{2}\left(\frac{e^x - e^{-x}}{2} - \frac{x^9}{9!} - \frac{x^7}{7!} - \frac{x^5}{5!} - \frac{x^3}{3!} - x\right), \qquad (16)$$

$P_4(x) = p_4x^4 + p_3x^3 + p_2x^2 + p_1x + p_0$ is a polynomial of degree three and d is a real number. Furthermore, from the results of [4] we have $\|\ell\|_{W_2^{(6,5)*}} = \|\psi_\ell\|_{W_2^{(6,5)}}$ and

$$\|\ell\|^2_{W_2^{(6,5)*}} = (\ell, \psi_\ell). \qquad (17)$$

Hence, taking into account equalities (7) and (15) we come to the following expression for the norm of ℓ:

$$\|\ell\|^2 = (\ell,\psi_\ell) = -\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_5[\beta]C_5[\gamma]G_1(h\beta - h\gamma) +$$
$$+2\sum_{\beta=0}^{N} C_5[\beta](\int_0^1 G_4'(x-h\beta)dx + \sum_{\gamma=0}^{N} C_0[\gamma]G_4'(h\beta-h\gamma) - \sum_{\gamma=0}^{N} C_1[\gamma]G_3(h\beta-h\gamma) - \sum_{\gamma=0}^{N} C_3[\gamma]G_2(h\beta-h\gamma))$$
$$+\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_0[\beta]C_0[\gamma]G_6(h\beta-h\gamma) - 2\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_0[\beta]C_1[\gamma]G_6'(h\beta-h\gamma) - 2\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_0[\beta]C_3[\gamma]G_5'(h\beta-h\gamma)$$
$$-\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_1[\beta]C_1[\gamma]G_5(h\beta-h\gamma) - 2\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_1[\beta]C_3[\gamma]G_4(h\beta-h\gamma) - \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_3[\beta]C_3[\gamma]G_3(h\beta-h\gamma)$$
$$-\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_1[\beta]C_1[\gamma]G_5(h\beta-h\gamma) - 2\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_1[\beta]C_3[\gamma]G_4(h\beta-h\gamma) - \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_3[\beta]C_3[\gamma]G_3(h\beta-h\gamma)$$
$$(18)$$

where $G_6(x)$ is defined by (16),

$$G_1(x) = \frac{\mathrm{sgn}x}{2}\left(\frac{e^x - e^{-x}}{2}\right), G_2(x) = \frac{\mathrm{sgn}x}{2}\left(\frac{e^x - e^{-x}}{2} - x\right), G_3(x) = \frac{\mathrm{sgn}x}{2}\left(\frac{e^x - e^{-x}}{2} - \frac{x^3}{3!} - x\right),$$
$$G_4(x) = \frac{\mathrm{sgn}x}{2}\left(\frac{e^x - e^{-x}}{2} - \frac{x^5}{5!} - \frac{x^3}{3!} - x\right), G_5(x) = \frac{\mathrm{sgn}x}{2}\left(\frac{e^x - e^{-x}}{2} - \frac{x^7}{7!} - \frac{x^5}{5!} - \frac{x^3}{3!} - x\right). \qquad (19)$$

Thus, we have calculated the norm of the error functional (7).

In the next section we find the minimum of the expression (18) by coefficients $C_5[\beta]$, $\beta = 0,1,...,N$, under the condition (13).

2.2 The minimization of the norm (18)

Here we solve the problem of finding the minimum of (18) by coefficients $C_5[\beta], \beta = 0,1,...,N$ under the condition (13). For this we use the Lagrange method.

Consider the following function

$$\Psi(C_5[0],C_5[1],...,C_5[N],d)=\|\ell\|^2-2d(\ell,e^{-x}).$$

Equating to zero the partial derivatives of the function Ψ by $C_5[\beta]$, $\beta=0,1,...,N$ and d we get the following system of $N+2$ linear equations

$$\sum_{\gamma=0}^{N}C_5[\gamma]G_1(h\beta-h\gamma)+de^{-h\beta}=F(h\beta),\ \ \beta=0,1,2,...,N, \tag{20}$$

$$\sum_{\gamma=0}^{N}C_5[\gamma]e^{-h\gamma}=g, \tag{21}$$

where

$$F(h\beta)=\int_0^1 G_4'(x-h\beta)dx+\sum_{\gamma=0}^{N}C_0[\gamma]G_4'(h\beta-h\gamma)-\sum_{\gamma=0}^{N}C_1[\gamma]G_3(h\beta-h\gamma)-\sum_{\gamma=0}^{N}C_2[\gamma]G_2(h\beta-h\gamma), \tag{22}$$

$$g=e^{-1}-1+\sum_{\gamma=0}^{N}C_0[\gamma]e^{-h\gamma}-\sum_{\gamma=0}^{N}C_1[\gamma]e^{-h\gamma}-\sum_{\gamma=0}^{N}C_2[\gamma]e^{-h\gamma}. \tag{23}$$

In this system $C_5[\beta],\beta=0,1,...,N$ and d are unknowns, that is, the above system has $N+2$ unknowns and $N+2$ linear equations. This system has only solution for every fixed natural N and this solution gives the minimum to the norm (18).

Further, we find an exact solution of the system (20)-(21).

2.3.The solution of the system (20)-(21).

In this section we solve the system (20)-(21). Here we use the concept of discrete argument functions (or functions of discrete argument) and operations on them following by S.L.Sobolev [7, 8].

Suppose φ and ψ are real-valued functions of real variable x and are defined in the real line $\mathbb{R}$. Let h be a small positive number.

A function $\varphi(h\beta)$ is called *a discrete argument function* if it is defined on some of integer values of β. *The inner product* of two discrete argument functions $\varphi(h\beta)$ and $\psi(h\beta)$ is defined as the following number

$$\left[\varphi(h\beta),\psi(h\beta)\right]=\sum_{\beta=-\infty}^{\infty}\varphi(h\beta)\cdot\psi(h\beta),$$

if the series on the right hand side of the last equality converges absolutely.

The convolution of two discrete argument functions $\varphi(h\beta)$ and $\psi(h\beta)$ is the following inner product

$$\varphi(h\beta)*\psi(h\beta)=\left[\varphi(h\gamma),\psi(h\beta-h\gamma)\right]=\sum_{\gamma=-\infty}^{\infty}\varphi(h\gamma)\cdot\psi(h\beta-h\gamma)$$

Moreover, we use the following discrete analogue of the differential operator $\frac{d^2}{dx^2}-1$ constructed in the work [2].

The discrete analogue $D_1(h\beta)$ of the differential operator $\frac{d^2}{dx^2}-1$ satisfying the equation

$D_1(h\beta)*G_1(h\beta)=\delta_d(h\beta)$

has the form

$$D_1(h\beta)=\frac{1}{1-e^{2h}}\begin{cases}0, & |\beta|\geq 2,\\ -2e^h, & |\beta|=1,\\ 2(1+e^{2h}), & \beta=0,\end{cases} \qquad (24)$$

where $G_1(h\beta)=\frac{\mathrm{sgn}(h\beta)}{2}\left(\frac{e^{h\beta}-e^{-h\beta}}{2}\right)$ and $\delta_d(h\beta)=\begin{cases}1, & \beta=0,\\ 0, & \beta\neq 0.\end{cases}$

Furthermore, $D_1(h\beta)$ has the following properties

$$D_1(h\beta)*e^{h\beta}=0 \text{ and } D_1(h\beta)*e^{-h\beta}=0. \quad (25)$$

Now suppose that $C_5[\beta]$=0 when $\beta=-1,-2,...$ and $\beta=N+1,N+2,....$ Then we can rewrite the system (20)-(21) in the following convolution form

$$C_5[\beta]*G_1(h\beta-h\gamma)+de^{-h\beta}=F(h\beta),\ \beta=0,1,...,N, \qquad (26)$$

$$\sum_{\gamma=0}^{N}C_5[\gamma]e^{-h\gamma}=g \qquad (27)$$

where

$$F(h\beta)=(e^{-h\beta}-e^{h\beta-1})\cdot\frac{e+1}{4}\left(1-\frac{h}{2}+\frac{h^2}{12}-\frac{h^4}{720}-\frac{h}{e^h-1}\right), \qquad (28)$$

and

$$g=\frac{1-e}{e}\cdot\left(1-\frac{h}{2}+\frac{h^2}{12}-\frac{h^4}{720}-\frac{h}{e^h-1}\right) \qquad (29)$$

which are obtained by calculating the right hand sides of (22) and (23), respectively.

We have the following main result of the work.

Theorem 2.1 *The coefficients of the optimal quadrature formula in the form (1) in the space $W_2^{(6,5)}(0,1)$ have the following forms:*

$$C_5[0]=\frac{h(e^h+1)}{2(e^h-1)}+\frac{h^4}{720}-\frac{h^2}{12}-1,$$

$$C_5[\beta]=0, \qquad \beta=1,2,...,N-1, \tag{30}$$

$$C_5[N]=-\frac{h(e^h+1)}{2(e^h-1)}-\frac{h^4}{720}+\frac{h^2}{12}+1.$$

Proof. We denote the left hand side of (26) by

$$u(h\beta)=C_5[\beta]*G_1(h\beta)+de^{-h\beta}. \tag{31}$$

Then we get

$$C_5[\beta]=D_1(h\beta)*u(h\beta). \tag{32}$$

Indeed, if the discrete argument function $u(h\beta)$ is defined at all integer values of β, then using (24) and (32), and taking (25) into account, we get

$$D_1(h\beta)*u(h\beta)=D_1(h\beta)*(G_1(h\beta)*C_5[\beta])+D_1(h\beta)*(d\ e^{-h\beta})$$

$$=C_5[\beta]*(D_1(h\beta)*G_1(h\beta))$$

$$=C_5[\beta]*\delta_d(h\beta)$$

$$=C_5[\beta].$$

Hence, in order to find $C_4[\beta]$ the function $u(h\beta)$ must be found at all integer values of β.

From (26) we get that

$$u(h\beta)=F(h\beta)\ for\ \beta=0,1,...,N, \tag{33}$$

where $F(h\beta)$ is defined by (28).

Next we find $u(h\beta)$ for $\beta=-1,-2,...$ and $\beta=N+1,N+2,...$.

For the cases $\beta=-1,-2,...$, from (31), using (27), we get

$$u(h\beta)=-\frac{1}{4}e^{h\beta}g+e^{-h\beta}\frac{1}{4}\sum_{\gamma=0}^{N}C_5[\gamma]e^{h\gamma}+de^{-h\beta}. \tag{34}$$

Similarly, for the cases $\beta=N+1,N+2,...$, we have

$$u(h\beta)=\frac{1}{4}e^{h\beta}g-e^{-h\beta}\frac{1}{4}\sum_{\gamma=0}^{N}C_5[\gamma]e^{h\gamma}+de^{-h\beta}. \tag{35}$$

Combining (33), (34) and (35), denoting $D=\frac{1}{4}\sum_{\gamma=0}^{N}C_5[\gamma]e^{h\gamma}$, we get the following

$$u(h\beta)=\begin{cases}-\frac{1}{4}e^{h\beta}g+(d+D)e^{-h\beta}, & \beta\le 0,\\ F(h\beta), & 0\le\beta\le N,\\ \frac{1}{4}e^{h\beta}g+(d-D)e^{-h\beta}, & \beta\ge N.\end{cases} \quad (36)$$

In the last equation d and D are unknowns. To find these unknowns we use the values of $u(h\beta)$ at points $\beta=0$ and $\beta=N$. Then we get the following system of equations

$$d+D-\frac{1}{4}g \quad = \quad F(0) \quad for\ \beta=0,$$

$$d-D-\frac{e^2}{4}g \quad = \quad eF(1) \quad for\ \beta=N.$$

Solving this system we get

$$d=0, \qquad D=\frac{e-1}{4}\left(1-\frac{h}{2}+\frac{h^2}{12}-\frac{h^4}{720}-\frac{h}{e^h-1}\right). \quad (37)$$

As a result, from (32) for $\beta=0,1,...,N$, using (24) and (36) with (37), by direct calculation we get (30). Theorem 2.1 is proved.

Thus, we have found the optimal coefficients $C_5[\beta]$, $\beta=0,1,2,...,N$ satisfying the equality (14).

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