Comparison of Finite-Difference Schemes for the Burgers Problem

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ABSTRACT

A large number of problems in physics and engineering leads to boundary value or initial boundary value problems for linear and nonlinear partial differential equations. At the same time, the number of tasks with analytical solutions is limited. These are problems in canonical domains, such as, for example, a rectangle, a circle, or a ball, and usually for equations with constant coefficients. In practice, it is often necessary to solve problems in very complex areas and for equations with variable coefficients, often nonlinear. This leads to the need to search for approximate solutions using various numerical methods.

KEYWORDS: *Lax method, Lax-Wendroff method, McCormack method, Warming—Cutler—Lomax method.*

Introduction

Naturally, the history of computational hydromechanics is closely related to the history of computer development. Until the end of World War II, most problems were solved by analytical and empirical methods. Until that time, only a few pioneers used numerical methods to solve problems. Calculations were performed manually and each individual solution was obtained as a result of a very large amount of work. Since computers were created, routine work related to obtaining results with numerical solution is carried out quite simply.

Modern problems of mathematical physics impose various requirements on the applied numerical algorithms[1], the main of which are; high order of approximation, stability of algorithms, conservativeness, monotony, cost-effectiveness, universality of algorithms, adaptation of algorithms, the possibility of parallelization of calculations This article describes and studies in detail various finite-difference schemes with which you can solve the simplest model equations. We will limit ourselves to considering the first-order wave equation. These equations are called model equations, since they are used to study the properties of solutions to more complex partial differential equations. Thus, the heat conduction equation can be considered as a model for other parabolic partial differential equations, for example, boundary layer equations. All considered model equations have analytical solutions under certain boundary and initial conditions. Knowing these solutions, it is easy to evaluate and compare various finite difference methods used to solve more complex partial differential equations. Of the many existing finite-difference methods for solving partial differential equations, this article describes mainly such methods that have properties characteristic of a whole class of similar methods. Some finite-difference methods useful for solving equations are not given, since they are similar to those described [2, 3].

For comparison, the most popular finite difference schemes were used, such as: the Lax method, the Lax—Wendroff method, the Mac-Cormack method, the Uorming- Cutler—Lomax method.

There are various finite-difference methods in the world and their application to solving simple linear problems. This allowed us to better understand these methods and get acquainted with their main specific features. But in hydromechanics, nonlinear problems usually have to be solved, since pressure, density, temperature and velocity must be determined from solving a nonlinear system of partial differential equations.

It is useful to first study the nonlinear Burgers equation [4]. It has the form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\mu\frac{\partial^2 u}{\partial x^2}. \quad (1)$$

The first and second terms in the left part of this equation are, respectively, non-stationary and convective terms, and in the right part there is a viscous term. If the viscous term is not zero, then equation (1) is parabolic; if it is zero, then only the non-stationary and nonlinear convective terms remain in the equation. Such an equation is hyperbolic and has the form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0. \quad (2)$$

Equation (2) can be considered as a model for Euler equations describing the motion of an ideal fluid. Equation (2) is a nonlinear convection equation and has some mathematical features, which are presented in the article. The article describes various difference schemes used to solve the inviscid Burgers equation. At the same time, typical results obtained in calculations using many widely used difference schemes will be presented, and the role of nonlinear terms will be clarified.

Equation (2) can also be interpreted as a nonlinear wave equation, while the wave propagation velocity at different points will be different. Since the velocity of propagation of perturbations changes, the characteristics begin to intersect and discontinuities appear in the solution, similar to shock waves in gas dynamics. Consequently, the one-dimensional model equation under consideration allows us to study the properties of discontinuous solutions.

Nonlinear hyperbolic partial differential equations have two types of solutions according to Lax [5]. Let 's explain this by the example of a simple scalar equation

$$\frac{\partial u}{\partial t}+\frac{\partial F}{\partial x}=0. \quad (3)$$

In general, both the unknown u and the function $F(u)$ are vectors. Let's rewrite equation (3) in the form

$$\frac{\partial u}{\partial t}+A\frac{\partial u}{\partial x}=0. \quad (3a)$$

where in the general case is the $A=A(u)$ — matrix Jacobi $\frac{\partial F_i}{\partial u_i}$, and in our simple example $A=\frac{\partial F}{\partial u}$. Since our partial differential equation (or system of equations) is hyperbolic, then all the eigenvalues of matrix A are real. A smooth solution of equation (3a) is called when the function and is continuous inside the domain, and its derivative may have a discontinuity at the boundary (the solution of the equation is Lipschitz continuous). A weak solution is a solution of equation (3a) that is smooth (x,t) everywhere except for some surface in space on which the function and can have a discontinuity. Certain restrictions are imposed on the magnitude of the jump of the function and when passing through the discontinuity surface. Let's return to the inviscid Burgers equation and find the conditions for the existence of a weak solution of this equation, i.e. the necessary conditions for the existence of a solution with a discontinuity, as shown in Fig. 1.

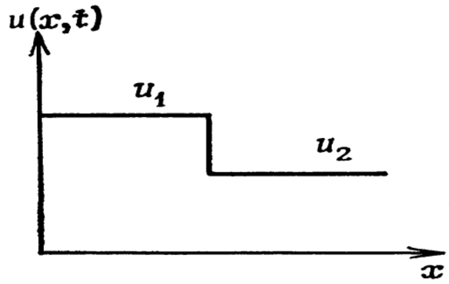

Fig. 1. A typical problem of the propagation of the gap for the Burgers equations.

DESCRIPTION OF THE SCHEME

Lax Scheme

The Lax method [5] was chosen as a typical method of the first order of accuracy in order to show that such methods allow solving nonlinear equations, but have strong dissipative properties.

$$u_i^{n+1}=\left(\frac{u_{i+1}^n-u_{i-1}^n}{2}\right)-\frac{\Delta t}{\Delta x}\frac{\left(F_{i+1}^n-F_{i-1}^n\right)}{2}. \qquad (4)$$

For the Burgers equation $F_i^n=u_i^n u_i^n/2$.

This is an explicit one - step scheme of the first order of accuracy with an approximation $O\left(\Delta t,(\Delta x)^2/\Delta t\right)$. It is stable at $\frac{\Delta t}{\Delta x}\le 1$.

The Lax - Wendroff method

The Lax-Wendroff method [6] is one of the first finite-difference methods of the second order of accuracy created for solving hyperbolic partial differential equations.

$$u_i^{n+1}=u_i^n-\frac{\Delta t}{2\Delta x}\left(F_{i+1}^n-F_{i-1}^n\right)+\frac{(\Delta t)^2}{2(\Delta x)^2}\left(A_{i+1/2}^n\left(F_{i+1}^n-F_i^n\right)-A_{i-1/2}^n\left(F_i^n-F_{i-1}^n\right)\right). \qquad (5)$$

The Jacobi matrix A is calculated in the middle between the nodes of the difference grid, i.e.

$$A_i=A\left(\frac{u_i+u_{i+1}}{2}\right).$$

If the Burgers equation is solved, then $F_i^n=u_i^n u_i^n/2$ and $A=u$. In this case

$$A_{i+1/2}=\left(\frac{u_i+u_{i+1}}{2}\right),\ A_{i-1/2}=\left(\frac{u_i+u_{i-1}}{2}\right).$$

This is an explicit one - step scheme of the second order of accuracy with an approximation $O\left((\Delta t)^2,(\Delta x)^2\right)$ stable at $\frac{\Delta t}{\Delta x}\le 1$.

The Mac-Cormack method

The Mac-Cormack method [7] is widely used to solve equations of gas dynamics. Mac-Cormack is especially convenient for solving nonlinear partial differential equations. Applying the explicit predictor-corrector method to the linear wave equation, we obtain the following difference scheme:

Predictor $$\overline{u_i^{n+1}} = u_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1}^n - F_i^n\right), \tag{6}$$

Proofreader $$u_i^{n+1} = \left(u_i^n + \overline{u_i^{n+1}} - \frac{\Delta t}{\Delta x}\left(\overline{F_i^{n+1}} - \overline{F_{i-1}^{n+1}}\right)\right). \tag{7}$$

This is an explicit one - step scheme of the second order of accuracy with an approximation $O\left((\Delta t)^2, (\Delta x)^2\right)$ stable at $\frac{\Delta t}{\Delta x} \le 1$.

The Uorming - Cutler - Lomax method

Uorming et al. [8] proposed a method of the third order of accuracy, which coincides in time with the Mac-Cormack method in the first two steps and with the Rusanov method in the third:

step 1 $$\overline{u_i^{n+1}} = u_i^n - c\frac{2\Delta t}{3\Delta x}\left(F_{i+1}^n - F_i^n\right), \tag{8}$$

step 2 $$\overline{\overline{u_i^{n+1}}} = \left(u_i^n + \overline{u_i^{n+1}} - c\frac{2}{3}\frac{\Delta t}{\Delta x}\left(\overline{F_i^{n+1}} - \overline{F_{i-1}^{n+1}}\right)\right). \tag{9}$$

step 3 $$u_i^{n+1} = u_i^n - 1/24c\frac{\Delta t}{\Delta x}\left(-2F_{i+2}^n + 7F_{i+1}^n - 7F_{i-1}^n + 2F_{i-2}^n\right) - -3/8c\frac{\Delta t}{\Delta x}\left(\overline{\overline{F_{i+1}^{n+1}}} - \overline{\overline{F_{i-1}^{n+1}}}\right) - \omega/24\left(u_{i+2}^n - 4u_{i+1}^n + 6u_i^n - 4u_{i-1}^n + u_{i-2}^n\right). \tag{10}$$

This is an explicit one - step scheme of the second order of accuracy with an approximation $O\left((\Delta t)^2, (\Delta x)^2\right)$ stable at $\frac{\Delta t}{\Delta x} \le 1$.

DISCUSSION OF THE RESULTS

The results of the calculation by the Lax method of the 1-0 gap moving to the right are shown in Fig.2.

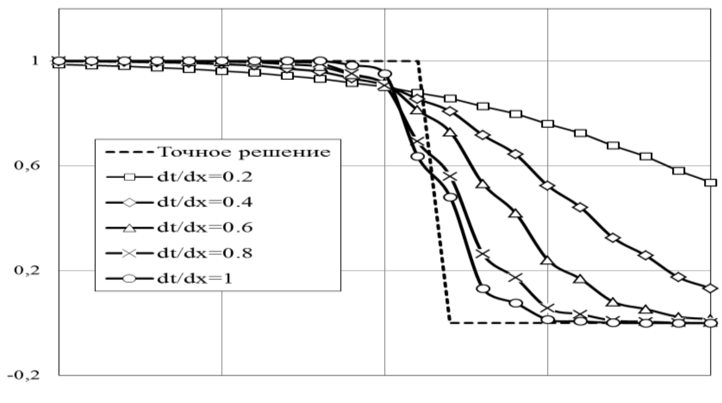

Fig. 2. Results of numerical solution of the Burgers equation according to the Lax scheme.

The position of the moving gap is determined quite accurately, but the dissipative properties of the method are manifested in smearing the gap into several steps of the difference grid. As noted earlier, this smearing should be the greater the smaller the number of the Clock. It is interesting to note that when calculating discontinuous solutions, the Lax method leads to the same values in two neighboring nodes, as shown in the figure. We will point out one more property of the Lax method — its monotonicity, i.e., the absence of oscillation of the solution. In the article [9] he showed that schemes with a higher order of accuracy than the first one cannot be monotonic.

Figure 3 shows the results of the calculation by the Lax—Wendroff method.

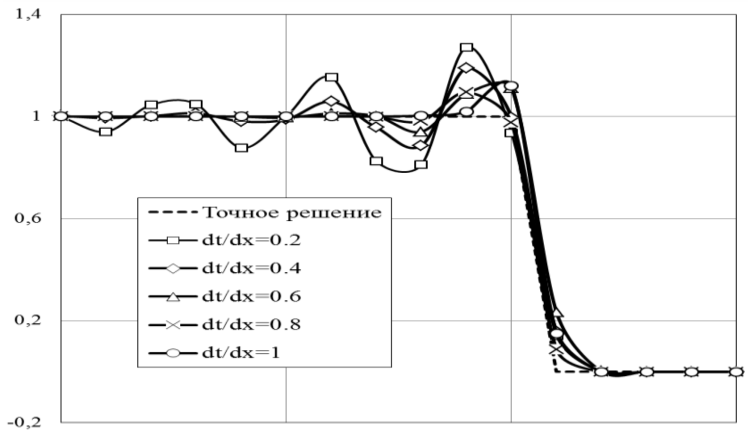

Fig. 3. The solution of the inviscid Burgers equation obtained by the Lax-Wendroff scheme.

The position of the gap moving to the right is determined quite accurately, and the gap itself is described by a rather steep line. The oscillations of the solution near the gap emphasize mainly the dispersion properties of the difference scheme. Although central differences are used to approximate the derivatives, the solution is asymmetric as the gap moves. With the number of Chimes equal to 0.2, stronger oscillations of the solution are observed than with the number of Chimes equal to 1. Usually, when the number of Chimes decreases, the quality of the numerical one deteriorates.

The results of calculating the right-moving gap by the McCormack method are shown in Fig. 4.

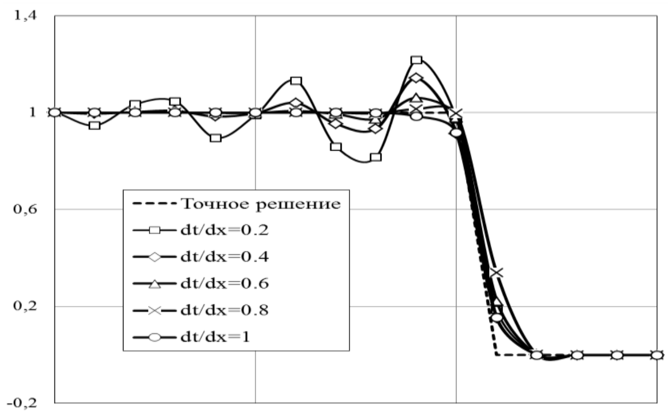

Fig. 4. The solution of the Burgers equation obtained by the Mac-Cormack scheme.

The position of the gap is determined quite accurately. The calculation results differ from those obtained by the Lax—Wendroff method at the same Courant numbers. This is a consequence of both a change in the direction of numerical differentiation at the predictor and corrector steps, and a consequence of the nonlinearity of the partial differential equation under consideration. It is not surprising that different results are obtained by methods that are equivalent for linear problems.

Usually, the McCormack scheme describes discontinuities very well. Note, among other things, that changing the direction of numerical differentiation at the predictor and corrector steps will lead to a change in the calculation results. Discontinuities are best calculated if, at the step, the difference predictor is taken in the direction of the gap movement.

Figure 5 schematically shows how when using the Uorming -- Cutler-Lomax method.

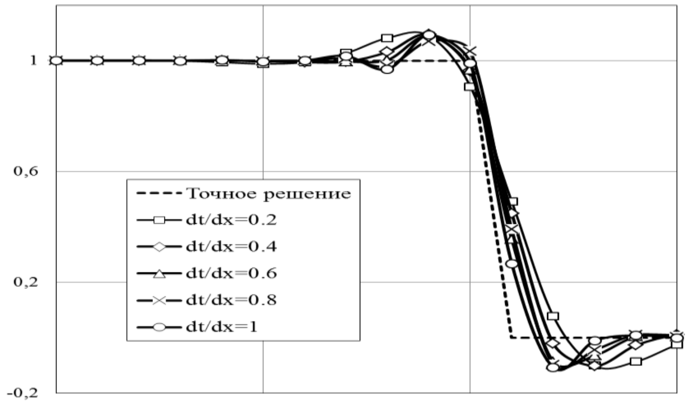

Fig. 5. The solution of the Burgers equation obtained by the Warming- Cutler-Lomax scheme.

Figure.5 shows the results of calculating this scheme for solving the Burgers equation with a right-moving gap. The position of the rupture and its intensity are described correctly, however, before and after the rupture, there is an excess of the exact values. Figure.5 schematically shows how, when using the Warming — Cutler—Lomax method, movement occurs along the points of the template when moving from one layer to another.

CONCLUSION.

In this article, we studied the basic finite-difference solution methods for simple model equations such as the Burgers equation. At the same time, the task was not set to describe all known methods for solving these equations. However, the presented methods are a reasonable prerequisite for analyzing methods for solving more complex problems [10-14]. From the information presented in the article, it can be seen that many different numerical methods can be used to solve the same problem. The difference in the quality of solutions obtained by these methods is often small, so it is quite difficult to choose the optimal method. However, it is possible to choose the best method using the experience gained during programming by various numerical methods and subsequent computer solution of model equations.

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