Differential LG - Game of Many Participant Players

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Abstract:

In this article, we have considered a simple motion differential game of pursuers and one evader in. Here controls of the pursuers are subjected to linear constraints which is the generalization of both integral and geometrical constraints, and control of the evader is subjected to a geometrical constraint. To solve a pursuit problem, the attainability domain of each pursuer has been constructed and therefore, necessary and sufficient conditions have been obtained by intersection of them.

Keywords: Differential game, evader, pursuer, strategy, geometric representation, integral constraint, attainability domain.

1. IntroductIon

Differential games were initiated by Isaacs [23]. Fundamental results were obtained by Bercovitz [6]–[7], Basar [4]-[5], Chikrii [9], Elliot and Kalton [10], Fleming [11]–[12], Friedman [13], Hajek [15]–[16], Ho, Bryson and Baron [17], Pontryagin [27], Krasovskii [24], Petrosjan [26], Pshenichnyi [28]–[29] and others. The book of Isaacs [23] contains many specific game problems that were discussed in details and proposed for further study. One of them is the “Life-line” problem, which rather was comprehensively studied by Petrosjan [26] by approximating measurable controls with most efficient piecewise constant controls that presents the strategy of parallel approach. Later this strategy was called Π-strategy. The strategies proposed in [1], [26], [28] for a simple motion pursuit game with geometrical constraints became the starting point for the development of the pursuit methods in games with multiple pursuers (see e.g. [2], [14], [30]). The problem is reduced to one pursuer and one evader problem subject to a state constraint. To prolong the capture time, a suboptimal control strategy for the evader was proposed. A relay-pursuit strategy was applied in [3], according to which only one pursuer is assigned to go after the evader at every instant of time.

At the present time there are more than a hundred monographs on the theory. Nevertheless, completely solved samples of Differential Games are quite few. This work is devoted to the Pursuit problem when linear constraints which are the generalization of integral as well as geometrical constraints are imposed on the pursuers’ control class and only a geometrical constraint is imposed on the evader’s control class and here it is studied in the term of winning of the pursuer.

2. STATEMENT OF THE PROBLEM

Consider the differential game when Pursuer $\mathbf{X}_i$, $i=1,2,...,m$ and Evader $\mathbf{Y}$. having radius vectors x_i and y correspondingly move in R^n. If their velocity vectors are u_i and v then the game will be described by the equations:

$$\dot{x}_i = u_i, \quad x_i(0) = x_{i0}, \qquad (2.1)$$

$$\dot{y} = v, \quad y(0) = y_0, \qquad (2.2)$$

where $x_i, y, u_i, v \in R^n$, $n \geq 1$ and x_{i0}, y_0 are the initial positions of the objects $\mathbf{X}_i$ and $\mathbf{Y}$. Here the temporal variation of u_i must be a measurable function $u_i(\cdot):[0,\infty) \to R^n$, we impose a constraint of the form

$$\int_0^t |u_i(s)|^2\, ds \leq L_i(t), \quad t \geq 0, \qquad (2.3)$$

admitting a linear change $L_i(t) = k_i t + \rho_{i0}$ in the time t, where k_i is arbitrary and ρ_{i0} is a nonnegative numbers. From the physical point of view, the right-hand of inequality (2.3) corresponds to the linear change of the given resource depending on the time $t \geq 0$. Therefore, the linear function $L_i(t)$ can be called a current change of the given resource of the Pursuers $\mathbf{X}_i$. Clearly, this resource increases if $k_i > 0$, decreases if $k_i < 0$ and remains unchanged if $k_i = 0$. In the last case, constraint (2.3) is called an Integral constraint. If $\rho_{i0} = 0$ and $k_i > 0$, then constraint (2.3) can be called a Geometrical constraint.

We call inequality (2.3) as L*–constraint* (Linear constraint) and denote by U_L^i the class of admissible controls, i.e., of all measurable functions satisfying the $L-$constraint.

Similarly, the temporal variation of v must be a measurable function $v(\cdot):[0,\infty) \to R^n$, and on this vector-function, we impose a geometrical constraint (briefly, $G-$constraint)

$$|v(t)| \leq \beta \quad \text{for almost every } t \geq 0, \qquad (2.4)$$

where β is a nonnegative parametric number which means the maximal velocity

of the evader. We denote by V_G the class of the evader's admissible controls satisfying constraint (2.4).

In the *LG*-game (2.1)–(2.4), the objective of the Pursuer $\mathbf{X}_i$ is to catch the Evader $\mathbf{Y}$, i.e., reach the equality $x_i(t) = y(t)$, where $x_i(t)$ and $y(t)$ are trajectories generated during the game. The notion of a "trajectories generated during the game" requires clarification. the Evader $\mathbf{Y}$ tries to avoid an encounter, and if it is impossible, postpone the moment of the encounter as far as possible. Naturally, this is a preliminary problem setting.

Here we are going to study mainly the game with phase constraints for the Evader being given by a subset A of R^n which is called the "Life-line" [23] (for the Evader naturally). Notice that in the case $A = \varnothing$ we have a simple *LG*-game .

Definition 1. *By the equations and initial conditions in (2.1)–(2.2), any pairs* $(x_{i0}, u_i(\cdot))$, $u_i(\cdot) \in U_L^i$ *and* $(y_0, v(\cdot))$, $v(\cdot) \in V_G$ *generate the trajectories*

$$x_i(t) = x_{i0} + \int_0^t u_i(s)\, ds, \qquad y(t) = y_0 + \int_0^t v(s)\, ds \qquad (2.5)$$

respectively. In this case, $x_i(t)$ *is called the pursuer's motion trajectory and* $y(t)$ *is called the evader's motion trajectory.*

Definition 2. *For each triple* $(k_i, \rho_{i0}, u_i(\cdot))$, $u_i(\cdot) \in U_L^i$ *the scalar quantity*

$$\rho_i(t) = L_i(t) - \int_0^t |u_i(s)|^2 \, ds, \quad \rho_i(0) = \rho_{i0}, \quad t \ge 0 \qquad (2.6)$$

is called the residual pursuer resource at a time moment t .

Definition 3. *The map* $u_i(\cdot): V_G \to U_L^i$ *is called the strategy for* $\mathbf{X}_i$ *if the following properties hold:*

1^o. *(Admissibility.) For every* $v(\cdot) \in V_G$, *the inclusion* $u_i(\cdot) = u_i[v(\cdot)] \in U_L^i$ *is valid.*

2^o. *(Volterranianity.) For every* $v_1(\cdot), v_2(\cdot) \in V_G$ *and* $t \ge 0$, *the equality* $v_1(s) = v_2(s)$ *a.e. on* $[0, t]$ *implies* $u_{i_1}(s) = u_{i_2}(s)$ *a.e. on* $[0, t]$ *with* $u_i(\cdot) = u_i[v(\cdot)] \in U_L^i$, $i = 1, 2, ..., m.$

Definition 4. *A strategy* $u_i(v)$ *is called winning for* $\mathbf{X}_i$ *on the interval* $[0,T]$ *in the LG-game, if for every* $v(\cdot) \in V_G$ *there exists a moment* $t^* \in [0,T]$ *that holds the equality* $x_i(t^*) = y(t^*)$.

Definition 5. *Assume that* $y_0 \notin \bar{A} \subset R^n$ *and a strategy* $u_i(v)$ *is winning for some* $\mathbf{X}_i$ *on the interval* $[0,T]$, *while* $\mathbf{Y}$ *stays in the zone* $R^n \setminus \bar{A}$ *i.e.,* $y_t(\cdot) = \{y(s): 0 \le s \le t\} \notin \bar{A}$ *and* $t \in [0,T]$. *Then the "Life-line" LG -game is called winning for the players* $\mathbf{X}_i$, $i = 1, 2, ..., m$ *on the interval* $[0,T]$.

Notice that $\bar{A}$ doesn't restrict any motion of $\mathbf{X}_i$ **.** Here $\bar{A}$ is the closure of the set $A \subset R^n$.

Let $x_{i0} \ne y_0$ and the current value of control $v(t),\ t \ge 0$ is given, where $v(\cdot) \in V_G$. Suppose that the triple (μ_{i0}, k_i, β) is a parametric state of the *LG*-game and denoted it by p_i. We find the following nonempty simply connected set of such states p_i

$$\mathbf{P}_{LG}^i = \mathbf{P}_1^i \cup \mathbf{P}_2^i \cup \mathbf{P}_3^i$$

where

$$\mathbf{P}_1^i = \{p_i : \mu_{i0} \ge 0, k_i > \beta^2, \beta \ge 0\},$$

$$\mathbf{P}_2^i = \{p_i : \mu_{i0} > 2\beta,\ k_i = \beta^2,\ \beta \ge 0\},$$

$$\mathbf{P}_3^i = \{p_i : \mu_{i0} \ge 2(\beta + \sqrt{\beta^2 - k_i}), k_i < \beta^2, \beta \ge 0\}$$

and $\mathbf{P}_1^i$, $\mathbf{P}_2^i$, $\mathbf{P}_3^i$ are mutually disjoint sets.

Definition 6. The function

$$u_i(v) = v - \lambda_i(v)\xi_{i0} \qquad (2.7)$$

is called *the strategy of parallel pursuit* (briefly, P_{LG}^i*-strategy*) for $\mathbf{X}_i$ in the *LG*-game,where $\lambda_i(v) = \mu_{i0}/2 + \langle v, \xi_{i0} \rangle + \sqrt{(\mu_{i0}/2 + \langle v, \xi_{i0} \rangle)^2 + k_i - |v|^2}$, $\xi_{i0} = z_{i0}/|z_{i0}|$, $z_{i0} = x_{i0} - y_0$, $\mu_{i0} = \rho_{i0}/|z_{i0}|$.

Property 1. *The scalar function* $\lambda_i(v)$ *is positively determined and continious in* $|v| \le \beta$ *for every*

$p_i \in \mathbf{P}_{LG}^i$.

Property 2. *If* $p_i \in \mathbf{P}_{LG}^i$, *then for* P_{LG}^i*-strategy, the equality*

$$|u_i(v)|^2 = k_i + \mu_{i0}\lambda_i(v). \qquad (2.8)$$

holds.

In [31], B.T.Samatov proved the following statements for the LG -game of one pursuer and one evader.

Theorem 1*. If in the LG -game,* $p \in \mathbf{P}_{LG}$ *and pursuer* $\mathbf{X}$ *apply* P_{LG}*-strategy then the equalities*

$$\rho(t) = \Lambda_{LG}(t, v(\cdot))\rho_0, \quad t \in [0, t^*], \qquad (2.9)$$

$$z(t) = x(t) - y(t) = \Lambda_{LG}(t, v(\cdot))z_0 \qquad (2.10)$$

hold, where $\Lambda_{LG}(t, v(\cdot)) = 1 - \frac{1}{|z_0|}\int_0^t \lambda(v(s))ds$ *is a scalar continuous monotone decreasing function with* t, $t \geq 0$, $t^* = min\{t : z(t) = 0\}$.

Theorem 2*. Let* $p \in \mathbf{P}_{LG}$ *in the LG –game. Then* P_{LG}*-strategy is winning on the time interval* $[0, T_{LG}]$*, where* $T_{LG} = |z_0| / \lambda_{LG}$ *and* $\lambda_{LG} = \mu_0 / 2 - \beta + \sqrt{(\mu_0 / 2 - \beta)^2 + k - \beta^2}$.

3. MaIn Results

Let the conditions of Theorem 1 and Theorem 2 hold. We suppose that in the moment t, $t ^3$ 0 the evader $\mathbf{Y}$ moves from a position y holding a constant vector v, $|v|£ b$. The pursuers $\mathbf{X}_i$ use Π_{LG}^i – strategy from a position x_i basing on the resource ρ_i, $\rho_i > 0$. Then w is a point where the pursuers $\mathbf{X}_i$ should meet the evader $\mathbf{Y}$. The set of all such points w will be given by relations:

$$|w - x_i| = T|u_i(v)|, \quad |w - y| = T|v|, \quad T|u_i(v)|^2 = k_i T + \rho_i, \quad T > 0$$

and from these relations we find

$$W_{LG}^i(x_i, y, \rho_i) = \left\{w : |w - x_i|^2 \geq (k_i / \beta^2)|w - y|^2 + (\rho_i / \beta)|w - y|\right\}. \qquad (3.1)$$

Theorem 3. *If* $p_i \in \mathbf{P}_{LG}^i$ *in the LG -game (2.1)-(2.4), then*

$$W_{LG}^i(t) = x_i(t) + \Lambda_i(t, v(\cdot))[W_{LG}^i(0) - x_{i0}], \quad t \in [0, t_i^*], \qquad (3.2)$$

where $W_{LG}^i(t) = W_{LG}^i(x_i(t), y(t), r_i(t))$, $t_i^* = \min\{t : z_i(t) = 0\}$, $i = 1, 2, ..., m$ *and*

$W_{LG}^i(0) = \left\{w : |w - x_{i0}|^2 \geq (k_i / \beta^2)|w - y_0|^2 + (\rho_{i0} / \beta)|w - y_0|\right\}$.

Proof. Consider the situation when $\mathbf{X}_i$ holds the Π_{LG}^i – strategy while $\mathbf{Y}$ on $[0, t_i^*]$ applies any control $v(\cdot) \in V_G$. Assume that $x_i(t)$, $y(t)$ are the current positions of players and $\rho_i(t)$ is the current resource of the pursuer. Then from (2.9) and (3.1) we have

$$W_{LG}^{i}(t)-x_i(t)=\left\{w:|w|^2\geq(k_i/\beta^2)\,|w+z_i(t)|^2+(\rho_i(t)/\beta)\,|w+z_i(t)|\right\}=$$

$$=\left\{w:|w|^2\geq(k_i/\beta^2)\,|w+\Lambda_i(t,v(\cdot))z_{i0}|^2+(\Lambda_i(t,v(\cdot))\rho_{i0}/\beta)\,|w+\Lambda_i(t,v(\cdot))z_{i0}|\right\}=$$

$$=\left\{w:|w/\Lambda_i(t,v(\cdot))|^2\geq(k_i/\beta^2)\,|w/\Lambda_i(t,v(\cdot))+z_{i0}|^2+\right.$$

$$\left.+(\rho_{i0}/\beta)\,|w/\Lambda_i(t,v(\cdot))+z_{i0}|\right\}=\Lambda_i(t,v(\cdot))[W_{LG}^{i}(0)-x_{i0}].$$

Property 3. *If* $p_i\in\mathbf{P}_{LG}^{i}$, *then for* $W_{LG}^{i}(t)$ *is convex set on* $[0,\,t_i^*]$.

Theorem 4. *Let* $p_i\in\mathbf{P}_{LG}^{i}$ *in the LG -game (2.1)-(2.4). Then the relation*

$$W_{LG}^{i}(t_2)\subset W_{LG}^{i}(t_1) \qquad (3.3)$$

is true for $\forall t_1,\,t_2\in[0,t^*],\ 0\leq t_1\leq t_2$.

P r o o f. From (2.8) we have

$$\left|v|z_{i0}|-\lambda_i(v)z_{i0}\right|^2=k_i\,|z_{i0}|^2+\lambda_i(v)\rho_{i0}\,|z_{i0}|\,.$$

Then from $|v(t)|\leq\beta$ and Property 1 we obtain

$$\left|(v|z_{i0}|/\lambda_i(v)+y_0)-x_{i0}\right|^2\geq(k_i/\beta^2)\left|(v|z_{i0}|/\lambda_i(v)+y_0)-y_0\right|^2+$$

$$+(\rho_{i0}/\beta)\left|(v|z_{i0}|/\lambda_i(v)+y_0)-y_0\right|$$

or

$$v|z_{i0}|/\lambda_i(v)+y_0\in W_{LG}^{i}(0) \qquad (3.4)$$

Using properties of the supporting function $F(W,p)=\sup_{w\in W}\langle w,p_i\rangle$ in [8] and from (3.4) we have

$$\langle v|z_{i0}|/\lambda_i(v)+y_0,\psi\rangle\leq F(W_{LG}^{i}(0),\,\psi)\Rightarrow$$

$$\Rightarrow\langle v,\psi\rangle-\frac{1}{|z_{i0}|}\lambda_i(v)F(W_{LG}^{i}(0)-y_0,\,\psi)\leq 0 \qquad (3.5)$$

for all $\psi\in R^n,\ |\psi|=1$. From (1.1), (1.7), (3.2) and (3.4) we have

$$F(W_{LG}^{i}(t),\psi)=\langle x_i(t),\psi\rangle+\Lambda_i(t,v(\cdot))F(W_{LG}^{i}(0)-x_{i0},\psi)\Rightarrow$$

$$\Rightarrow\frac{d}{dt}F(W_{LG}^{i}(t),\psi)=\langle\dot{x}_i(t),\psi\rangle-\frac{1}{|z_{i0}|}\lambda_i(v)F(W_{LG}^{i}(0)-x_{i0},\psi)=$$

$$=\langle v-\lambda_i(v)\xi_{i0},\psi\rangle-\frac{1}{|z_{i0}|}\lambda_i(v)F(W_{LG}^{i}(0)-x_{i0},\psi)=$$

$$=\langle v,\psi\rangle-\frac{1}{|z_{i0}|}\lambda_i(v)F(W_{LG}^i(0)-y_0,\psi)\le 0.$$

Property 4. *If* $p_i\in\mathbf{P}_{LG}^i$, *then an inclusion* $y(t)\in W_{LG}^i(0)$ *is valid on the time interval* $t\in[0,t_i^*]$.

Theorem 5. *If in the LG -game* $p_i\in\mathbf{P}_{LG}^i$ *hold for some* $i=1,2,...,m$, *then* $y(t)\in W_{LG}$ *on the time interval* $[0,T_{LG}]$, *where* $W_{LG}=\bigcap_{i=1}^{m}W_{LG}^i(0)$, $T_{LG}=d/\beta$, $d=\max\{|w_1-w_2|:\ w_1,w_2\in W_{LG}\}$.

P r o o f. This follows from Theorems 2-4 and Property 4.

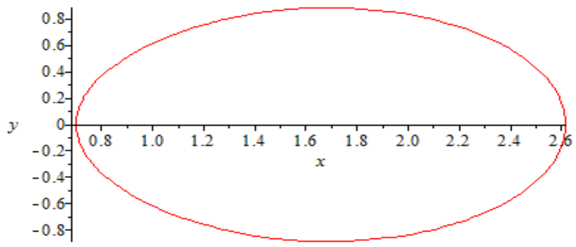
Theorem 6. *If* $W_{LG}\cap A=\varnothing$, *then in the "Life-line" LG –game, the players* $\mathbf{X}_i$, $i=1,2,...,m$ *win on the interval* $[0,T_{LG}]$.

P r o o f. This follows from Theorem 5.

4. GEOMETRICAL REPRESENTATIONS

We will show geometrical representation of the set boundary *of attainability points of the LG -game for some i=1,2…,m in some particular cases in* R^2.

1. Let $p_i\in\mathbf{P}_1^i$, $x_{i0}=(0,0)$, $y_0=(1,0)$, $k_i=2$, $\beta=1$, $\rho_{i0}=1$, for some then its equation $x_i^2+y^2=2((x_i-1)^2+y^2)+\sqrt{(x_i-1)^2+y^2}$ and its smooth line consists of Cartesian's oval (Picture-1).

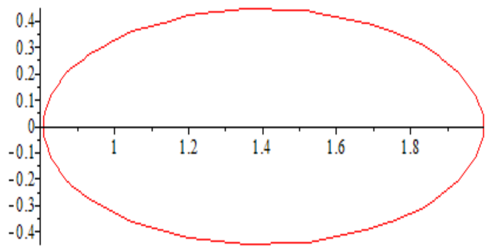


Picture-1

2. If $p_i\in\mathbf{P}_2^i$, $x_{i0}=(0,0)$, $y_0=(1,0)$, $\beta=1$, $\rho_{i0}=3$, then its equation

$$(25/9)\cdot(x_i-7/5)^2+5y^2=1$$

and its smooth line consists of Ellipse (Picture-2).

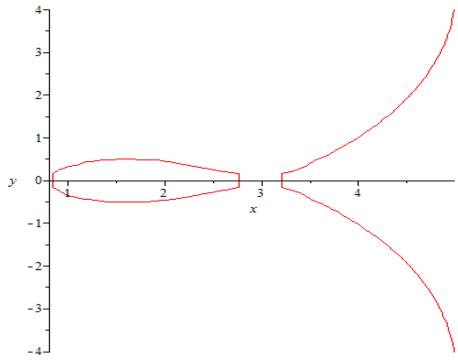


Picture-2

3. If $p_i \in \mathbf{P}_3^i$, $x_{i0} = (0,0)$, $y_0 = (1,0)$, $k_i = \frac{3}{4}$, $\beta = 1$, $\rho_{i0} = 3$, then its equation

$$x_i^2 + y^2 = \frac{3}{4}((x_i - 1)^2 + y^2) + 3\sqrt{(x_i - 1)^2 + y^2}$$

and its smooth line consists of the inner loop of Pascal's snail (Picture-3).



Picture-3

References

1. Azamov A. (1986) On the quality problem for simple pursuit games with constraint. Serdica Bulgariacaemath. Publ.Sofia: 12(1): 38–43.
2. Azamov A.A., Samatov B.T. (2010) The -Strategy: Analogies and Applications. The Fourth International Conference Game Theory and Management, St.Petersburg: 33–47.
3. Bakolas E., Tsiotras P. (2011) On the relay pursuit of a maneuvering target by a group of pursuers. In 50th IEEE Conference on Decision and Control and European Control Conference, Orlando, FL, pp. 4270–4275.
4. Basar T. Stochastik differential games and intricacy of information structures. Dynamic Games in Economics.- Springer, 2014.- P.23-49.
5. Basar T., Bernhard P. H-infinity optimal control and related mini-max design problems: a dynamic game approach. - Boston: Birkhauser, 1995.- P. 428.
6. Berkovitz L.D. (1967) A Survey of Differential Games, Mathematical Theory of Control. New York, Academic Press 373–385.
7. Berkovitz L.D. (1986) Differential game of generalized pursuit and evasion. SIAM J. Contr.: 24(3): 361–373.
8. Blagodatskikh V.I. (2001). Introduction to optimal control. Linear theory. Moscow. Graduate School.
9. Chikrii A.A. (1997) Conflict-Controlled Processes. Kluwer, Dordrecht.
10. Elliot R.J., Kalton N.J. (1972) The Existence of Value for Differential Games. American Mathematical Soc.
11. Fleming W.H. (1957) A note on differential games of prescribed duration. Contributions to the Theory of Games. 3: 407–416.
12. Fleming W. H. (1961) The convergence problem for differential games. J. Math. Anal. Appl. 3: 102–116.

13. Friedman A. (1971) Differential Games. Wiley Interscience, New York.

14. Grigorenko N.L. (1990). Mathematical Methods of Control for Several Dynamic Processes. Izdat. Gos. Univ., Moscow.

15. Hajek O. (1975). Pursuit games. New York, Academic Press.

16. Hajek O. (2008) Pursuit Games: An Introduction to the Theory and Applications of Differential Games of Pursuit and Evasion, Dove. Pub. New York.

17. Ho Y., Bryson A., Baron S. (1965) Differential games and optimal pursuit-evasion strategies. IEEE Trans Autom Control 10: 385-389.

18. Ibragimov G.I. (1998). On the optimal pursuit game of several pursuers and one evader. Prikladnaya Matematika i Mekhanika. 62(2): 199–205.

19. Ibragimov G.I. (2005). Optimal pursuit with countable many pursuers and one evader, Differential Equations, 41(5): 627–635.

20. Ibragimov G.I. (2013). The optimal pursuit problem reduced to an infinite system of differential equations. J. Appl. Maths Mekhs. 77(5): 470–476.

21. Ibragimov G.I. (2013). Optimal pursuit time for a differential game in the Hilbert space *l*2. ScienceAsia, 39S: 25– 30.

22. Ibragimov G.I., Abd Rasid N., Kuchkarov A.Sh. and Ismail F. (2015) Multi pursuer differential game of optimal approach with integral constraints on controls of players. Taiwanese Journal of Mathematics, 19(3):963–976, Doi: 10.11650/tjm.19.2015.2288.

23. Isaacs R. (1965) Differential games. John Wiley and Sons, New York.

24. Krasovskii N.N., Subbotin A.I. (1974) Positional Differential Games. Nauka, Moscow. (in Russian)

25. Pashkov A.G. and Terekhov S.D. (1987) A differential game of approach with two pursuers and one evader. Journal of Optimization Theory and Applications, 55(2): 303–311.

26. Petrosjan L.A. (1993). Differential games of pursuit. Series on optimization, Vol.2. World Scientific Poblishing, Singapore.

27. Pontryagin L.S. (2004) Selected Works. MAKS Press, Moscow.

28. Pshenichnyi B.N. (1976). Simple pursuit by several objects. Cybernetics and System Analysis. 12(3): 484-485. DOI 10.1007/BF01070036.

29. Pshenichnyi B.N., Chikrii A.A., and Rappoport J.S. (1981) An efficient method of solving differential games with many pursuers, Dokl. Akad. Nauk SSSR 256, 530–535. (in Russian).

30. Samatov B.T. (2013) On a Pursuit-Evasion Problem under a Linear Change of the Pursuer Resource. Siberian Advances in Mathematics, Allerton Press, Inc.Springer. New York: 23(4): 294–302.

31. Samatov B.T. (2014) The Π-strategy in a differential game with linear control constraints. J. Appl. Maths and Mechs, Elsevier. Netherlands. 78(3): 258–263.