Application of the derivative to solving equations and proving inequalities.

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Abstract. *This article shows examples of solving problems using a derivative. Examples are considered with the use of the derivative in solving equations and proving inequalities, as well as solving problems with parameters.*

In conclusion, the article provides several rules for the application of the derivative.

Keywords. *Derivative, differential, interval, mean inequality, continuous, increasing, decreasing.*

Introduction. The field of application of the derivative in solving problems of mathematics is very wide. This is, for example, the use of the derivative when forming algebraic expressions, factoring, proving identities, calculating sums, solving equations, inequalities and systems, proving inequalities, solving problems with parameters, investigating functions, etc.

Here we restrict ourselves to the derivative method to solve the equations and prove the inequalities.

The application of the derivative to the solution of equations, inequalities and systems, as well as to the proof of inequalities, is based on the connection between the increase of decrease of a function on a certain interval and the sign of its derivative.

Example – 1. Solve the equation

$$x^2-2x+2=2\sqrt[4]{2x-1}-x$$

Solution. Rewriting this equation in the form

$$x^2-2x+2=2\sqrt{2x-1}-x \qquad (1)$$

Note that its ruts are the abscissas of the points of intersections or tangency of the graphs of the function $f(x)=x^2-2x+2$ and $g(x)=2\sqrt[4]{2x-1}-x$. To find out the relative position of the graphs of these functions, we will find their extremum points.

Since $f(x)=(x-1)^2+1$, this function reaches its smallest value, equal to 1, at the point $x=1$. The domain of existence of the function $g(x)=2\sqrt[4]{2x-1}-x$ consists of all x such that $x\ge\frac{1}{2}$. As

$$g'(x)=2\cdot\frac{1}{4}(2x-1)^{-3/4}\cdot 2-1=\frac{1-(2x-1)^{3/4}}{(2x-1)^{3/4}},\ \ x>\frac{1}{2},\ \text{then}$$

$g'(x)>0$ at $1/2<x<1$,

$g'(x)=0$ at $x=1$,

$g'(x)<0$ at $x>1$.

Since the function $y=g(x)$ is continuous on. $[1/2+\infty)$, we conclude from that the function $g(x)$ increases on the interval $[1/2,1]$ and decreases on the interval $[1;+\infty)$. Therefore, the point $x=1$ is the largest value of the function $g(x)$ on its domain of existence. Thus, for any $x\in[1/2+\infty)$

$x^2-2x+2\geq 1,$

$2\sqrt[4]{2x-1}-x\leq 1$

Therefore, equation (1) and, therefore, the original equation has a single root $x=1$.

Example – 2. For each value *A*, find the number of foots of the equation

$$x^2-3x^2-A=0 \qquad (2)$$

Solution. Let us find the sections of increase and decrease of the function $f(x)=x^3-3x^2$. Since $f'(x)=3x^2-6x=3x(x-2)$, then

$f'(x)<0$ at $0<x<2$,

$f'(x)=0$ at $x=0$ and $x=2$

$f'(x)>0$ at $x<0$ and $x>2$

Thus, a continuous function $y=f(x)$ at the point $x=2$ has a local minimum, and at the point $x=0$ it has a local maximum, and $f(0)=0,\ f(2)=-4$.

In addition, the function $y=f(x)$ decreases on the interval $[0;2]$ and increases on the intervals $(-\infty;0]$ and $[2;+\infty)$. Moreover, $\lim\limits_{x\to+\infty} f(x)=+\infty$ and $\lim\limits_{x\to-\infty} f(x)=-\infty$.

Hence it follows that to find out the dependence of the number of roots of equation (2) on the possible values of *A*, it is necessary to find out the relative position of the graph of the function $y=f(x)$ and the straight line $y=A$, when *A* changes in the $(-\infty;+\infty)$. From the properties of the function $f(x)$ is continuous at every point of its domain of existence and is a polynomial of the third degree, which means that it has either one real root or three real roots, we conclude that (fig.1).

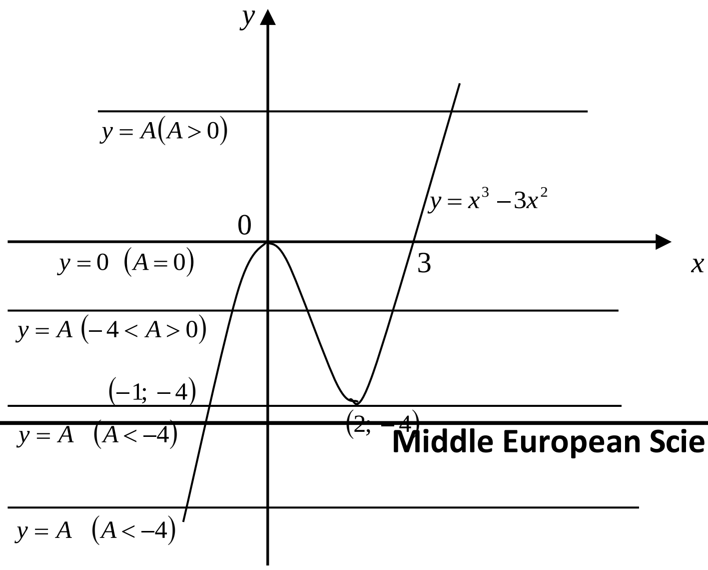

fig. 1.

for $A > 0$, the equation has one root for $A = 0$ the equation has three roots, $(x_1 = x_2 = 0,\ x_3 = 3)$ among which two coinciding, for $-4 < A < 0$ the equation three different roots, for $A = -4$ the equation has three roots $(x_1 = x_2 = 2,\ x_3 = -1)$ among which two coinciding; for $A < -4$ the equation has one root. If $x_1, x_2, \ldots, x_n$ the roots of the algebraic equation

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0 \quad a_0 \neq 0, \qquad (3)$$

then the polynomial on the left side of this equation can be represented as

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 = a_n(x - x_1)(x - x_2)\ldots(x - x_n).$$

If among the roots $x_1, x_2, \ldots, x_n$ there are roots equal to each other, then they say that equation (3) has multiple roots, while if $x_1 = x_2 = \ldots = x_n = A$ and among the other roots there is no equal to the number *A*, they say that number $x = A$ is a root of multiplicity *k*. For example since

$$x^4 - 7x^3 + 9x^2 + 27x - 54 = (x-3)^3 \cdot (x+2),$$

then the root $x = 3$ has multiplicity 3, and the root $x = -2$ is a root of multiplicity 1 (or a simple root).

The use of the derivative allows, without solving equation (3), not only to verify the existence of multiple roots (if they are) but also to give a way to select all multiple roots by separating them from simple roots. The following statement holds:

The greatest common divisor of the polynomials $f(x)$ and $f'(x)$ has as their roots only the roots of the polynomial $f(x)$, and only those of them that have multiplicity at least 2. Each of these multiple roots of $f(x)$ is the root of the greatest common divisor of multiplicity by one below. Simple roots of $f(x)$ are not the roots of the greatest common divisor of polynomials $f(x)$ and $f'(x)$.

This implies the following rule for finding multiple roots of equation (3):

1. Find $f'(x)$.
2. Find the greatest common divisor of the polynomials $f(x)$ and $f'(x)$.
3. Find the roots of the greatest common divisor of the polynomials $f(x)$ and $f'(x)$.

Each of the found roots of the greatest common divisor of the polynomials $f(x)$ and $f'(x)$ is a polynomial $f(x)$, and the multiplicity in this root is one greater than its multiplicity in the greatest common divisor.

Note that if the greatest common divisor of the polynomials $f(x)$ and $f'(x)$ is a constant, then the equation $f(x) = 0$ has no multiple roots.

Example – 3. Solve the equation

$$x^3 - 8x^2 + 13x - 6 = 0$$

Solution. Consider the polynomial

$$f(x) = x^3 - 8x^2 + 13x - 6,$$

whose derivative is

$$f'(x) = 3x^2 - 16x + 13$$

Find the greatest common divisor of the polynomials $f(x)$ and $f'(x)$.

We have

$$\begin{array}{l|l}
x^3 - 8x^2 + 13x - 6 & \underline{3x^2 - 16x + 13} \\
\underline{-x^3 - \frac{16}{3}x^2 + \frac{13}{3}x} & \frac{1}{3}x - \frac{8}{9} \\
\quad -\frac{8}{3}x^2 + \frac{26x}{3} - 6 & \\
- \quad \underline{-\frac{8}{3}x^2 + \frac{128x}{9} - \frac{104}{9}} & \\
\quad -\frac{50}{9}x + \frac{50}{9} = -\frac{50}{9}(x-1) &
\end{array}$$

$$\begin{array}{l|l}
3x^2 - 16x + 13 & \underline{x-1} \\
\underline{3x^2 - 3x} & 3x - 13 \\
\quad -13x + 13 & \\
\quad \underline{-13x + 13} & \\
\qquad 0 &
\end{array}$$

Thus, the greatest common divisor of the polynomials $f(x)$ and $f'(x)$ is $x-1$. Since $x=1$ is a simple root of the greatest common divisor, then the number $x=1$ will be a two-fold root of this equation, and, therefore, the polynomial $f(x)$ is divisible without a remainder by $(x-1)^2$.

Dividing $f(x)$ by $(x-1)^2$, we find that $f(x) = (x-1)^2 \cdot (x-6)$. Therefore, the roots of the original equation are the numbers $x_1 = x_2 = 1$ and $x = 6$, and only they.

Example – 4. Find all functions $f(x)$, each of which has a continuous derivative for all $x \in R$ and satisfies the identity $f(2x) = 2f(x),\ x \in R$.

Solution. Let $f(x)$ - be the required function. Then from the identity $f(2x) = 2f(x)$ we have $f(0) = 0$ and $2f'(2x) = 2f'(x)$, whence we find

$$f'(2x) = f'(x)$$

Since *x* is an arbitrary number, we get

$$f'(2x) = f'(x) = f'\left(\frac{x}{2}\right) = f'\left(\frac{x}{4}\right) = \ldots = f'\left(\frac{x}{2^n}\right) = \ldots \qquad x \in R$$

Since $\frac{x}{2^n} \to 0$ for $n \to \infty$ and by the condition $y = f'(x)$ is a continuous function on *R*, then

$$f'(x) = \lim_{n\to\infty} f'(x) = \lim_{n\to\infty} f'\left(\frac{x}{2^n}\right) = f'\left(\lim_{n\to\infty} \frac{x}{2^n}\right) = f'(0),\ x \in R$$

Thus, the function $f'(x),\ x \in R,$ is a constant, whence $f(x) = Ax + B,$ where $A = f'(0)$. Setting, for example, $x = 0,$ from the last identity, taking into account the equality $f(0) = 0$, we obtain that $B = 0$. Therefore, the required functions can only be functions of the form $f(x) = Ax$. By checking, we make sure that any function of the form $f(x) = Ax$ satisfies the condition of the problem.

Example – 5. Prove that inequality

$$a^3 + b^3 + c^3 \geq 3abc$$

holds for any positive values a, b and c.

Solution. Without loss of generality, we can assume, for example, that $0 < a \leq b \leq c$. Consider the function

$$f(x) = x^3 + b^3 + c^3 - 3xbc$$

since $f'(x) = 3x^2 - 3bc = 3(x^2 - bc),$

then $f'(x) < 0$ for $0 < x < b \neq c$. Hence it follows that the function $f(x)$ decreases on the interval $[0; b]$. In this way $f(x) \geq f(b),$ i.e.

$$x^3 + b^3 + c^3 - 3xbc \geq 2b^3 + c^3 - 3b^2c$$

Consider now the function

$$g'(x) = 6x^2 - 6xc = 6x(x - c)$$

Hence $g'(x) < 0$ at $0 < x < c$, and therefore, the function $g(x)$ decreases on the interval $[0; c]$. We conclude that $g(b) \geq g(c)$, i. e.

$$2b^3 + c^3 - 3b^2c \geq 2c^3 + c^3 - 3c^2c = 0$$

This completes the proof of the required inequality.

Example – 6. Prove that for positive numbers $x_1, x_2, \ldots, x_n$ not exceeding 1, the inequality

$$(1 + x_1)^{1/x_2} \cdot (1 + x_2)^{1/x_3} \cdot (1 + x_3)^{1/x_4} \cdot \ldots \cdot (1 + x_n)^{1/x_1} \geq 2^n \text{ and } c.$$

Solution. Lemma $(1 + x)^n \geq 1 + nx$ for $n > 1,\ x > -1$ (Bernoulli inequality). To prove the lemma, it suffices to note that the function $f(x) = (1 + x)^n - 1 - nx$ has a minimum at the point $x = 0$, since $f'(x) = n(1 + x)^{n-1} - n < 0$ for $-1 < x < 0$ and $f'(x) > 0$ for $x > 0$, and $f(0) = 0$. The lemma is proved by virtue of the lemma

$$(1 + x_1)^{1/x_2} \geq 1 + \frac{x_1}{x_2},\ (1 + x_2)^{1/x_3} \geq 1 + \frac{x_2}{x_3}, \ldots, (1 + x_n)^{1/x_1} \geq 1 + \frac{x_n}{x_1}$$

and the mean inequality

$$(1 + x_1)^{1/x_2} \cdot (1 + x_2)^{1/x_3} \cdot (1 + x_3)^{1/x_4} \cdots (1 + x_n)^{1/x_1} \geq \left(1 + \frac{x_1}{x_2}\right) \cdot \left(1 + \frac{x_2}{x_3}\right) \ldots \left(1 + \frac{x_n}{x_1}\right) \geq$$

$$\geq 2\sqrt{\frac{x_1}{x_2}} \cdot 2\sqrt{\frac{x_2}{x_3}}, \ldots, 2\sqrt{\frac{x_n}{x_1}} = 2^n$$

Example – 7. Prove that

$$\ln n \leq 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1}, \quad n \geq 2.$$

Solution. Let us prove an inequality of a more general nature, which also has other applications.

Let $y = f(x)$ have a convex upward derivative on some interval $(a;b)$, and the numbers $x_1, x_2, \ldots, x_n$ form this interval are numbers of arithmetic progression with the difference *d*. Then

$$f(x_n) \le f(x_1) + d\left(f'(x_1) + f'(x_2) + \ldots + f'(x_{n-1})\right) \qquad (*)$$

Indeed, since the function $y = f(x)$ is convex upward, its graph lies below the tangent drawn at any point $(x_k; f(k))$ of the graph of the function $y = f(x)$; in particular

$$f(x_{k+1}) \le f(x_k) + f'(x_k)(x_{k+1} - x_k), \quad k = 1,2,\ldots,n-1$$

those

$$f(x_{k+1}) \le f(x_k) + df'(x_k), \quad k = 1,2,\ldots,n-1.$$

Adding all the inequalities obtained for

$k = 1,2,3,\ldots,n-1$, we find that

$$f(x_n) \le f(x_k) + d\left(f'(x_1) + f'(x_2) + \ldots + f'(x_{k-1})\right).$$

If now $f(x) = \ln x,\ x_k = k\ (k = 1,2,\ldots,n)$, then it follows from (*) that

$$f(n) \le f(1) + f'(x_1) + f'(x_2) + \ldots + f'(x_{n-1}),$$

those

$$\ln n \le 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1}, \quad n \ge 2$$

Example – 8. Prove that

$$4tg5^0 tg9^0 < 3tg6^0 tg10^0..$$

Solution. Consider the function

$$f(x) = \frac{tgx}{x}, \ 0 < x < \frac{\pi}{4}.$$

As

$$f'(x) = \frac{x - \sin x \cos x}{x^2 \cos^2 x}, \ 0 < x < \frac{\pi}{4},$$

$$x - \sin x \cos x = \frac{1}{2}(2x - \sin 2x) > 0, \ 0 < x < \frac{\pi}{4},$$

then the function $f(x)$ increases on the interval $\left(0; \frac{\pi}{4}\right)$.

In this way,

$$\frac{tg\frac{5\pi}{180}}{\frac{5\pi}{180}} < \frac{tg\frac{6\pi}{180}}{\frac{6\pi}{180}}, \quad \frac{tg\frac{9\pi}{180}}{\frac{9\pi}{180}} < \frac{tg\frac{10\pi}{180}}{\frac{10\pi}{180}}$$

and hence,

$$4tg5^0 9^0 < 3tg6^0 tg10^0.$$

Example – 9. Which of the numbers is greater than, $\cos 2020$ or it $\cos 2021$

Solution. Consider the function $f(x) = x + \cos x$

Since $f'(x) = 1 - \sin x \geq 0$ and $f'(x) = 0$ for $x = \frac{\pi}{2} + 2\pi n,\ n \in Z$, then the function $y = f(x)$ increases on the set of all real numbers. Therefore $f(2020) < f(2021)$

those $\cos 2020 < 1 + \cos 2021$

Conclusion

The above solved problems are based on a number of basic theorems.

1. If a function $f(x)$ and $g(x)$ differentiable on the interval $(a;b)$ and $f(x) = g(x),\ x \in (a;b,)$, then $f'(x) = g'(x),\ x \in (a;b)$.
2. If a function $f(x)$ and $g(x)$ differentiable on the interval $(a;b)$ and $f'(x) = g'(x)$, then $f(x) = g(x) + c$, where c – some constant.
3. If a function $f(x)$ differentiable on the interval $(a;b)$ and $f'(x) = 0,\ x \in (a,b)$, the function $f(x)$ on the interval $(a;b)$ is identically equal to a constant, i.e. $f(x) = c$.
4. If a function $f(x)$ differentiable on the interval $(a;b)$ and $f'(x) > 0,\ (f'(x) < 0),\ x \in (a,b)$, then the function $f(x)$ increases (decreases) on the interval $(a;b)$.

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