Some vector equality and application to the solution of geometric problems

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Abstract. This article discusses three basic relationships and shows them when applied to solving geometric problems. However, teaching students how to use vectors to solve problems in a limited curriculum is difficult. To overcome these difficulties, you need a well-thought-out exercise system. The proposed article describes the experience in solving this issue.

Key words. Collinear, noncollinear, coplanar, centroid, tetrahedron.

Introduction

The apparatus of vector algebra, studied in the course of mathematics, is widely used in proving theorems and soling many geometric problems. However, teaching students how to use vectors to solve problems in a limited curriculum is difficult. To overcome these difficulties, you need a well-thought-out exercise system.

Let us now proceed to consider the basic vector relations and their application to solving problems.

1-basic ratio. Any triangle *ABC* satisfies the equality

$$\overline{MA}+\overline{MB}+\overline{MC}=\overline{0},$$

Where *M* is the centroid of triangle *ABC*. Let us prove the relation 1. Let *M* be the centroid of triangle *ABC* (fig.1). Let's connect point *M* with all the vertices of the triangle. Line *MB* intersects *AC* of triangle *ABC* at point *D*, which is the midpoint of side *AC*.

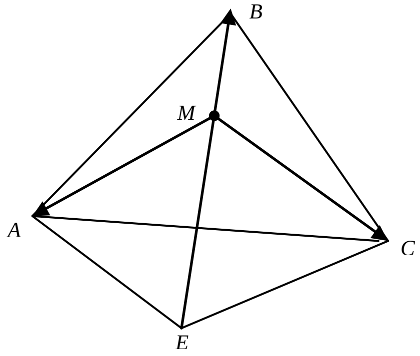


fig. 1.

On the straight line *BM* we put $ME=BM$ and connect the point *E* with the vertices *A* and *C*. Obviously, *AMCE* is a parallelogram. Therefore $\overline{ME}=\overline{MA}+\overline{MC}$, since $\overline{MB}=-\overline{ME}$, then $\overline{MA}+\overline{MB}+\overline{MC}=\overline{0}$. Let us show the application of relation 1 to problem solving.

Task 1.

The intersection points of the medians of triangles ABC and $A_1B_1C_1$ coincide. Prove that vectors AA_1, BB_1 and CC_1 are coplanar.

Solution. Based on 1 we have:

$$\overline{MA}+\overline{MB}+\overline{MC}=\overline{0} \qquad (1)$$

$$\overline{MA_1}+\overline{MB_1}+\overline{MC_1}=\overline{0} \qquad (2)$$

Subtracting parts from (1) and (2), we get:

$$\overline{AA_1}+\overline{BB_1}+\overline{CC_1}=\overline{0} \qquad (3)$$

Equality (3) means that vectors AA_1, BB_1 and CC_1 are coplanar.

Task 2. Prove that if *M* is the ctntroid of triangle *ABC* and *O* is an arbitrary point in space, then the equality

$$\overline{OM}=\frac{1}{3}\left(\overline{OA}+\overline{OB}+\overline{OC}\right) \qquad (4)$$

Evidence. We write the following vector equalities:

$$\overline{OM}+\overline{MA}=\overline{OA},$$

$$\overline{OM}+\overline{MB}=\overline{OB},$$

$$\overline{OM}+\overline{MC}=\overline{OC}.$$

Adding these equalities by parts, we get:

$$3\overline{OM}+\left(\overline{MA}+\overline{MB}+\overline{MC}\right)=\overline{OA}+\overline{OB}+\overline{OC},$$

from where

$$\overline{OM}=\frac{1}{3}\left(\overline{OA}+\overline{OB}+\overline{OC}\right)$$

2-basic ratio. Point *D* is taken in triangle *ABC* on side *AC* so that $AD;DC=m:n$. Then the following relation holds: (fig.2)

$$\overline{BD}=\frac{n}{m+n}\overline{BA}+\frac{m}{m+n}\overline{BC} \qquad (5)$$

Evidence. From triangle *ABC* we have:

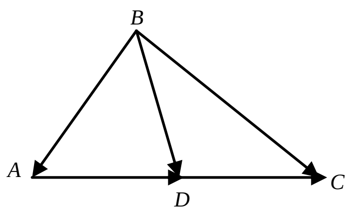


fig. 2

$$\overline{AC}=\overline{BC}-\overline{BA},$$

$$\overline{AD}=\frac{m}{m+n}\overline{AC}=\frac{m}{m+n}\overline{BC}-\frac{m}{m+n}\overline{BA},$$

$$\overline{BD}=\overline{BA}+\overline{AD}=\overline{BA}+\frac{m}{m+n}\overline{BC}-\frac{m}{m+n}\overline{BA}=$$

$$=\frac{n}{n+m}\overline{BA}+\frac{m}{m+n}\overline{BC}$$

Let us show further the application of (s) to the solution of problems.

Task 2. *A* line *AE* is drawn through the middle *E* of the median CC_1 of triangle *ABC*, intersecting side *BC* at point *F* (fig. 3). Calculate $AE;EF$ and $CF:FB$.

Solution. Introduce vectors $\overline{AB}=\overline{b}$ and $\overline{AC}=\overline{c}$. Let $CF:FB=m:n$. The by formula (5) we have:

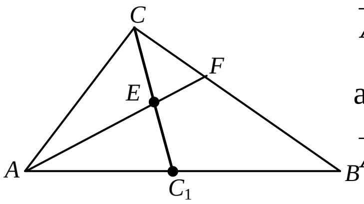


fig. 3

$$\overline{AF}=\frac{m}{m+n}\bar{b}+\frac{n}{m+n}\bar{c}$$

and

$$\overline{AF}=x\cdot\overline{AF}=\frac{xm}{m+n}\bar{b}+\frac{xn}{m+n}\bar{c} \qquad (6)$$

where $0<x<1$.

On the other hand, given that E is the midpoint of the median CC_1, we obtain the following expression for *AE*:

$$\overline{AE}=\frac{1}{2}\overline{AC}+\frac{1}{2}\overline{AC_1}=\frac{1}{2}\bar{c}+\frac{1}{4}\bar{b} \qquad (7)$$

Due to the uniqueness of the expansion of the vector in two no collinear vectors from (6) and (7), we obtain the system

$$\begin{cases}\dfrac{xm}{m+n}=\dfrac{1}{4},\\ \dfrac{xn}{m+n}=\dfrac{1}{2}\end{cases} \qquad (8)$$

Dividing by parts the first equation of system (8) by the second, we obtain that $m:n=1:2$, i.e. $CF:FB=1:2$. Adding the equations of system (8) by parts, we find that $x=\frac{3}{4}$, i.e. $AF:EF=3:1$.

3-basic ratio. Given *a* tetrahedron *ABCD* and point *M* in the plane of its face *ABC*. Prove that the decomposition $\overline{DM}=\alpha\overline{DA}+\beta\overline{DB}+\gamma\overline{DC}$ satisfies the equality

$$\alpha+\beta+\gamma=1$$

Evidence. Suppose that point *M* lies inside the triangle *ABC* (fig. 4). Draw a straight line through points *A* and *M* that intersects side *BC* at point *E*. let point *E* divide side *BC* in the ratio $m:n$, i.e. $BE:EC=m;n$. Then by formula (5)

$$\overline{DE}=\frac{m}{m+n}\overline{DC}+\frac{n}{m+n}\overline{DB}$$

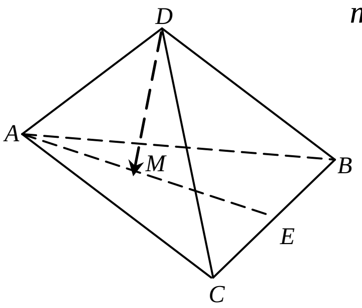


fig. 4

Let further point *M* divide segment *AE* in segment *AE* in the ratio $p:q$, i.e. $AM:ME=p;q$.

Then

$$\overline{DM}=\frac{p}{p+q}\overline{DE}+\frac{q}{p+q}\overline{DA}=$$

$$=\frac{p}{p+q}\left(\frac{m}{m+n}\overline{DC}+\frac{n}{m+n}\overline{DB}\right)+\frac{q}{p+q}\overline{DA}=$$

$$=\frac{q}{p+q}\overline{DA}+\frac{p}{p+q}\cdot\frac{n}{m+n}\overline{DB}+\frac{p}{p+q}\cdot\frac{m}{m+n}\overline{DC}$$

So, the vector $\overline{DM}$ is decomposed into vectors $\overline{DA}$, $\overline{DB}$ and $\overline{DC}$. It is easy to make sure that the sum of the coefficients in this decomposition is equal to 1.

I.e.

$$\frac{q}{p+q}+\frac{p}{p+q}\cdot\frac{n}{m+n}+\frac{p}{p+q}\cdot\frac{m}{m+n}=\frac{q}{p+q}+\frac{p}{p+q}\left(\frac{n}{m+n}+\frac{m}{m+n}\right)=\frac{q}{p+q}+\frac{p}{p+q}=1$$

Remarks. 1) In those cases when the point *M* lies outside the triangle or on one of its sides, the proofs are similar. 2) The proved relation is a necessary and sufficient condition for (A, B, C and *M*) to the same plane.

When solving geometric problems using the vector method, it is necessary to move from the geometric formulation of the problem to its vector description. Then, using the properties of vectors and operations on then, find some vector relationships that reflect the data and conditions of the problem, from which it is possible to obtain a solution to the problem.

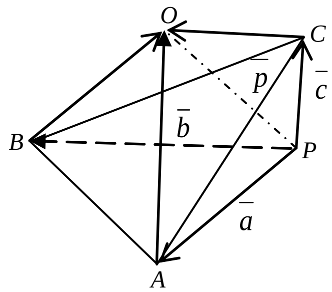
Given the lengths of three edges *PA*, *PB* and *PC* of the tetrahedron *PABC*, emanations from its vertex *P*, and the values of the plane angles at this vertex are also known, then using the vectors it is possible to find the radius, and therefore the area of the sphere (the volume of the sphere), described around this tetrahedron.

Consider the following tasks.

Task 3. In the triangular pyramid *PABC*, all planar angles at the vertex *P* are straight. Find the area of a sphere circumscribed about this pyramid if $PA=2,\ PB=3,\ PC=4$.

Solution. Let point *O* be the center of a sphere circumscribed about *a* tetrahedron *PABC*, *R* is the radius of the this sphere. Then $OA=OB=OC=OP=R$.

We introduce non-coplanar vectors $\overline{PA}=\overline{a},\ \overline{PB}=\overline{b},\ \overline{PC}=\overline{c}$ we will take them as basic in space. Then $\overline{PO}=x\overline{a}+y\overline{b}+z\overline{c}$ and $\left|\overline{PO}\right|=R$. Find the coefficients *x*, *y* and *z* in this expansion of the vector $\overline{PQ}$.



According to the triangle rule, we have:

$$\overline{p}=\overline{a}+\overline{AO}=\overline{b}+\overline{BO}=\overline{c}+\overline{CO}$$, whence

$$\overline{AO}=\overline{p}-\overline{a}\ ,\ \overline{BO}=\overline{p}-\overline{b},\ \ \overline{CO}=\overline{p}-\overline{c}.$$

From the equalities $OA=OB=OC=OP$ (as the radii of a sphere circumscribed about the *PABC* tetrahedron) it

fig. 5

follows that $\left|\overline{PO}\right|=\left|\overline{AO}\right|=\left|\overline{BO}\right|=\left|\overline{CO}\right|$, means

$$\overline{AO}^2=\overline{BO}^2=\overline{CO}^2=\overline{PO}^2=\overline{p}^2$$

Then we get

$$\begin{cases}\overline{p}^2-\overline{AO}^2=0\\ \overline{p}^2-\overline{BO}^2=0\\ \overline{p}^2-\overline{CO}^2=0\end{cases}\Leftrightarrow\begin{cases}\left(\overline{p}-\overline{AO}\right)\left(\overline{p}+\overline{AO}\right)=0\\ \left(\overline{p}-\overline{BO}\right)\left(\overline{p}+\overline{BO}\right)=0\\ \left(\overline{p}-\overline{CO}\right)\left(\overline{p}+\overline{CO}\right)=0\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}\left(\overline{p}-\left(\overline{p}-\overline{a}\right)\right)\left(\overline{p}+\left(\overline{p}-\overline{a}\right)\right)=0\\ \left(\overline{p}-\left(\overline{p}-\overline{b}\right)\right)\left(\overline{p}+\left(\overline{p}-\overline{b}\right)\right)=0\\ \left(\overline{p}-\left(\overline{p}-\overline{c}\right)\right)\left(\overline{p}+\left(\overline{p}-\overline{c}\right)\right)=0\end{cases}\Leftrightarrow\begin{cases}\overline{a}\cdot\left(2\overline{p}-\overline{a}\right)=0\\ \overline{b}\cdot\left(2\overline{p}-\overline{b}\right)=0\\ \overline{c}\cdot\left(2\overline{p}-\overline{c}\right)=0\end{cases}\Leftrightarrow$$

$$\Leftrightarrow \quad \begin{cases} \bar{a}\cdot\bar{p} = 0{,}5a^{-2} \\ \bar{b}\cdot\bar{p} = 0{,}5b^{-2} \\ \bar{c}\cdot\bar{p} = 0{,}5c^{-2} \end{cases}$$

Note that since the basic vectors $\bar{a}, \bar{b}, \bar{c}$ are pairwise perpendicular and their lengths are 2, 3 and 4, respectively, then

$$\bar{a}\cdot\bar{b} = \bar{a}\cdot\bar{c} = \bar{b}\cdot\bar{c} = 0,\ \ \bar{a}^2 = 4,\ \bar{b}^2 = 9,\ \bar{c}^2 = 16 \qquad (9)$$

Replacing P with the expression $x\bar{a} + y\bar{b} + z\bar{c}$ in the last system of equations and taxing into account (9), we get:

$$\begin{matrix} \bar{a}(x\bar{a} + y\bar{b} + z\bar{c}) = 0{,}5a^{-2} \\ \bar{b}(x\bar{a} + y\bar{b} + z\bar{c}) = 0{,}5b^{-2} \\ \bar{c}(x\bar{a} + y\bar{b} + z\bar{c}) = 0{,}5c^{-2} \end{matrix} \quad \Leftrightarrow \quad \begin{cases} x\bar{a}^2 = 0{,}5\bar{a}^2 \\ y\bar{b}^2 = 0{,}5\bar{b}^2 \\ z\bar{c}^2 = 0{,}5\bar{c}^2 \end{cases} \quad \Leftrightarrow \quad \begin{cases} x = 0{,}5 \\ y = 0{,}5 \\ z = 0{,}5 \end{cases}$$

Then

$$\overline{PQ} = 0{,}5\bar{a} + 0{,}5\bar{b} + 0{,}5\bar{c}$$

$$\overline{PQ}^2 = 0{,}25\bar{a}^2 + 0{,}25\bar{b}^2 + 0{,}25\bar{c}^2 + \bar{a}\cdot\bar{b} + \bar{b}\cdot\bar{c} + \bar{a}\cdot\bar{c} =$$

$$= 0{,}25(4 + 9 + 16) = \frac{29}{4},\quad \left|\overline{PQ}\right| = \frac{\sqrt{29}}{2}$$

Means, $S_{sphere} = 4\pi R^2 = 4\pi\cdot\frac{29}{4} = 29\pi$

The vector solution of many geometric problems is much simpler than their solution by means of elementary geometry, the reason for this simplification is that with the vector method of solving it is possible to do without those, additional constructions that should be performed in a purely geometric solution of even simple problems. Solving geometric problems, it is necessary to be able to translate the condition of a geometric problem into vector terminology and symbolism (into vector language), then perform the appropriate algebraic operations on vectors and, finally, translate the result obtained in vector form back into geometric language.

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