Collective Modes of Anyons Localized in 2D Anisotropic Harmonic Potential

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ABSTRACT

The goal of our research is to investigate the collective modes of the anyons localized in 2D parabolic well and get the right exact expression for their collective mode frequencies. A study of the collective motion of atomic gases, localized in the harmonic trap, belongs to class of actual and interesting problems of physics of ultra-cold atomic and molecular systems. A topology of 2D systems allows to exist the particles, whoes statistics may be orbitrary between bosons and fermions, therefore they call anyons. And these anyons are described with parameter and may be determined in the interval between ones for bosons and fermions. The one of intriguing problem of ultra-cold atomic gases is a study of the role of the anyon statistics to the system centre mass mode frequencies.

KEYWORDS: *2D, Anisotropic.*

1. INTRODUCTION

It is well-known that all elementary particles fall into one of two possible categories - bosons and fermions, depending on whether they obey the Bose-Einstein or the Fermi-Dirac statistics respectively. These particle are at least in 3-dimensional space-time. However in two space dimensions we do not have only bosons and fermions, but also particles with any statistics in between. These particles are called anyons and are the subject of this work.

Certainly, it is unusual feature of anyons that they arise only in two-dimensional systems and it is hard to imagine for both physicists working at totally different field of the physics and people far away from science these amazing particles. However, these particles are not simply topological fantasies or objects of purely mathematical interest; on the contrary they might play an important role in certain physical phenomena of the real world. Of course, since we are living in at least three space dimensions where particles can be only bosons or fermions, anyons are not real particles. However there exist certain condensed-matter systems (for example thin layers at the interface between different semiconductors) that can be regarded effectively as two-dimensional. Their localized excitations (if they exist) are quasi-particles subject to the rules of a two-dimensional world. It is these quasiparticles that may be anyons and may be observed in certain cases. For example the collective excitations above the ground state of systems exhibiting the fractional quantum Hall effect (for a review see (Prange and Girvin 1990)) have been identified as localized quasi-particles of fractional charge (Laughlin 1983), fractional spin and fractional statistics (Arovas et al. 1984; Halperin 1984), and thus they can be regarded as anyons. Furthermore, anyons are conjectured to play a role also in the theory of high temperature superconductivity (Chen et al. 1989), even though in this case no conclusive word can be said at the moment (Lyons et al. 1990; Kiefl et al. 1990; Spielman et al. 1990).

Such anyonic particles are becoming of increasing importance in condensed matter physics and quantum computation. They may play an essential role for describing the fractional quantum Hall effect, high-temperature superconductivity, and the physics of topological insulators and superconductors. Moreover, anyons as unusual quasiparticles with properties of its statistics are adequate tool for implementing a topological quantum computer. All of the mentioned categories in physics are the active issues of the physicists throughout the world. Certainly, all of these emphasize the importance of investigating the quantum anyon systems and their collective motion in the harmonic trap that has been done in this scientific work.

Most of the great interest that anyons have attracted in the past few years derives from the (unexpected) applications of these ideas to certain two-dimesional condensed matter systems, most notably those exhibiting the fractional quantum Hall effect (see for instance (Prange and Girvin 1990)). In this case a series of new states of matter emerge as incompressible quantum liquids (Laughlin 1983) around which the low-energy excitations are localized quasi-particles with unusual fractional quantum numbers, i.e. anyons. Furthermore, it is also very likely that anyonic excitations with fractional statistics exist in films of liquid 3He in the A-phase (Volovik and Yakovenko 1989). The application of anyons to the theory of high temperature superconductivity has also been considered quite extensively (for reviews see (Wilczek 1990; Lykken et al. 1991)), but their actual relevance in this context is quite controversial and doubtful.

Since experimentally first ultracold atoms have been realized in harmonic potentials, the goal of our work will be the consideration of the collective motion of anyons in the 2D harmonic trap.

The present paper is organized as follows. We start with introducing Hamiltonian of the anyons in 2D parabolic harmonic well in section2. Then, cumulant method is introduced in section3. In sections 4 and 5 some calculations have been given by utilizing this method. Next section is devoted for deriving the harmonic oscillator equation for the centre-of-mass - the main result of our work. Finally, at the end the conclusion is presented.

2. HAMILTONIAN OF ANYONS TRAPPED IN 2D ANISOTROPIC HARMONIC POTENTIAL

In this section we describe the Hamiltonian of anyons, localized in the 2D anisotropic trap, which expression will be taken from the paper [26] and, following to the paper of Ghost and Sinha [27], we write this system Langrangian.

The Hamiltonian of the gas of N anyons with mass m and charge e, confined in 2D parabolic well, is:

$$\hat{H} = \frac{1}{2m}\sum_{k=1}^{N}\left(\overrightarrow{p_k} + A_\nu(\overrightarrow{r_k})\right)^2 + \sum_{k=1}^{N}\frac{m}{2}\left(\omega_x^2 x_k^2 + \omega_y^2 y_k^2\right).$$

Here $\overrightarrow{r_k}$ and $\overrightarrow{p_k}$ represent the position and momentum operators of the k th anyon in 2D space dimension,

$$A_\nu(\overrightarrow{r_k}) = \hbar\nu\sum_{j\neq k}\frac{\overrightarrow{e_z}\times r_{\vec{k}j}}{\left|r_{kj}\right|^2}$$

is the anyon gauge vector potential [28], $\vec{r}_{kj} = \overrightarrow{r_k} - \overrightarrow{r_j}$ and $\overrightarrow{e_z}$ is the unit vector normal to the 2D plane. In the expression for vector potential $A_\nu(\overrightarrow{r_k}), \nu$ is the anyon factor and hereafter we assume that $0 \leq \nu \leq 1$, which means the variation of the anyon factor between bosonic and fermionic limits of anyons.

Our interest is the solution of the Schrödinger equation

$$i\hbar\frac{\partial\psi(\vec{R},t)}{\partial t} = \hat{H}\psi(\vec{R},T)$$

Let us consider first the term in the Hamiltonian $\hat{H}$, containing only the anyon vector potential $A_\nu(\overrightarrow{r_k})$. In the bosonic representation of anyons we take the system wave function in the form [29,30] :

$$\psi(\vec{R},t)=\prod_{k=j} r_{kj}^{\nu}\Psi_T(\vec{R},t)$$

Here and above $\vec{R}=\{\vec{r_1},\vec{r_2},\dots,\vec{r_n}\}$ is the configuration space of the N anyons. The product in the right hand side of this equation is the Jastrow-type wave function. It describes the short distance correlations between two particles due to anyonic (fermionic) statistics interaction.

By substituting the wave function of this form into Schrödinger equation (3) without the harmonic potential term, we obtain the equation:

$$i\hbar\frac{\partial\Psi_T(\vec{R},t)}{\partial t}=\left(\hat{H}_1+\hat{H}_2\right)\Psi_T(\vec{R},t)$$

where

$$\hat{H}_1=\sum_{k=1}^{N}\left(-\frac{\hbar^2\Delta_k}{2m}-\frac{\hbar^2\nu}{m}\sum_{j\neq k}\frac{\vec{r}_{kj}\cdot\vec{\nabla}_k}{\left|\vec{r}_{kj}\right|^2}\right)$$

and

$$\hat{H}_2=-i\frac{\hbar}{m}\sum_{k=1}^{N}\left(\vec{A}_\nu(\vec{r_k})\cdot\vec{\nabla}_k+\nu\sum_{j\neq k}\frac{\vec{A}_\nu(\vec{r}_k)\cdot\vec{r}_{kj}}{\left|\vec{r}_{kj}\right|^2}\right)$$

ing

As in the paper [27] of Ghosh and Sinha, by introduc-

$$a_0=\sqrt{\frac{\hbar}{m\omega_0}},$$

where $\omega_0=\sqrt{\omega_x\omega_y}$, we make dimensionless the length quantities and denote them by tilde sign.

We express the energy quantities in the Hamiltonian (1) in the units of $\hbar\omega_0$. Then, for instance, the harmonic potential term will have the form:

$$\frac{m}{2}\sum_{k=1}^{N}\left(\omega_x^2x_k^2+\omega_y^2y_k^2\right)=\frac{\hbar\omega_0}{2}\sum_{k=1}^{N}\left(\lambda\tilde{x_k}^2+\frac{1}{\lambda}\tilde{y_k}^2\right)$$

where $\lambda=\omega_x/\omega_y,\tilde{x}_k=x_k/a_0,\tilde{y}_k=y_k/a_0$ and parameter λ is the anisotropic parameter for the harmonic potential.

Now we make dimensionless Hamiltonians $\hat{H}_1$ and $\hat{H}_2$

$$\tilde{\hat{H}}_1=\sum_{k=1}^{N}\left(-\frac{\hbar^2\Delta_k}{2m}-\frac{\hbar^2\nu}{m}\sum_{j\neq k}\frac{\vec{r}_{kj}\cdot\vec{\nabla}_k}{\left|\vec{r}_{kj}\right|^2}\right)=$$
$$-\hbar\omega_0\sum_{k=1}^{N}\left(\frac{\tilde{\Delta}_k}{2}+\nu\sum_{j\neq k}\frac{\tilde{\vec{r}}_{kj}\cdot\tilde{\vec{\nabla}}_k}{\left|\tilde{\vec{r}}_{kj}\right|^2}\right)$$

since $\Delta_k=\partial^2/\partial x_k^2+\partial^2/\partial y_k^2=\tilde{\Delta}_k/a_0^2$.

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Similarly

$$\tilde{\hat{H}}_2 = -i\hbar\omega_0 \sum_{k=1}^{N} \left(\tilde{\vec{A}}_\nu(\tilde{\vec{r}}_k) \cdot \tilde{\vec{\nabla}}_k + \nu \sum_{j\neq k} \frac{\tilde{\vec{A}}_\nu(\tilde{\vec{r}}_k) \cdot \tilde{\vec{r}}_{kj}}{\left|\tilde{\vec{r}}_{kj}\right|^2} \right)$$

where

$$\tilde{\vec{A}}_\nu(\tilde{\vec{r}}_k) = \nu \sum_{j\neq k} \frac{\vec{e}_z \times \tilde{\vec{r}}_{kj}}{\left|\tilde{\vec{r}}_{kj}\right|^2}$$

And finaly, we obtain the dimensionless Schrödinger equation:

$$i\frac{\partial \Psi_T(\tilde{\vec{R}}, \tilde{t})}{\partial \tilde{t}} = \left(\tilde{\hat{H}}_1 + \tilde{\hat{H}}_2 + \frac{1}{2}\sum_{k=1}^{N} \left(\lambda \tilde{x_k}^2 + \frac{1}{\lambda}\tilde{y}^2 \right) \right) \Psi_T(\tilde{\vec{R}}, \tilde{t})$$

with $\tilde{t} = \omega_0 t$.

At the end of this section, we emphasize that the wave function $\Psi_T(\tilde{\vec{R}}, \tilde{t})$ contains the configurational space of N anyons vector $\vec{R}$. Therefore, it corresponds to many particle wave function of system. Previously, at the calculation of time variation of BEC, the wave function was a function of only one coordinate of condensate (see, for example, the paper [27]) and the solution of problem of BEC collective motions in the harmonic trap was essentially easier.

3. CUMULANTS EQUATION OF MOTION METHOD

For the description of above mentioned monope and quadrupole modes and also the oscillation of the centre of mass motion (the Kohn theorem), we use the cumulants equation of motion method [31, 34]. According to this method, for the small amplitude oscillations, it is convenient to take the trial many body wave function $\Psi_T(\vec{R}, t)$ in the Gaussian form (we use notations, taken from Ref. [31], for variational parameters):

$$\Psi_T(\vec{R}, t) = \left(\frac{1}{\pi q_1 q_2}\right)^{\frac{N}{2}} \prod_{k=1}^{N} \exp\left[-\left(\frac{1}{2q_1^2} + iA_1\right) \times (x_k - x_0)^2 + ix_k C_1 - \left(\frac{1}{2q_2^2} + iA_2\right)(y_k - y_0)^2 + iy_k C_2 \right]$$

Here, x_k and y_k are the x and y coordinate components of k-th particle, all variational parameters q_1, q_2, A_1, A_2 and C_1, C_2, and centre of mass components x_0 and y_0 are the time t dependent.

In order to derive the comulants equation of motion, we average over the Schrödinger equation (11) the weight $f_{x,y}^{i}$:

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$$i\int\prod_{k=1}^{N}dx_kdy_kf_{x,y}^{i}\Psi_T^*\frac{\partial\Psi_T}{\partial\tilde{t}}=\int\prod_{k=1}^{N}dx_kdy_kf_{x,y}^{i}\times$$
$$\Psi_T^*\left(\tilde{\hat{H}}_1+\tilde{\hat{H}}_2+\frac{1}{2}\sum_{k=1}^{N}\left(\lambda\tilde{x}_k^2+\frac{1}{\lambda}\tilde{y}_k^2\right)\right)\Psi_T$$

Since the wave function Ψ_T, Eq. (12), is a Gaussian, no zero these averages are only for two central moments. Averages with $f_x^1=x_k-x_0$ and $f_y^1=y_k-y_0$ provide equations to find the centre of mass motion. And averages with $f_x^2=(x_k-x_0)^2-q_1^2$ and $f_y^2=(y_k-y_0)^2-q_2^2$ provide equations to find the widths motion.

4. AVERAGE QUANTITIES FOR $f_{x,y}^1$ OF IDEAL GAS OF PARTICLES IN 2D ANISOTROPIC HARMONIC POTENTIAL

For the ideal gas of particles in 2D anisotropic harmonic potential, we have an averaged Schrödinger equation:

$$i\int\prod_{k=1}^{N}dx_kdy_kf_{x,y}^{1}\Psi_T^*\frac{\partial\Psi_T}{\partial\tilde{t}}=\int\prod_{k=1}^{N}dx_kdy_kf_{x,y}^{1}\times$$
$$\Psi_T^*\frac{1}{2}\sum_{k=1}^{N}\left(-\tilde{\Delta}_k+\lambda\tilde{x}_k^2+\frac{1}{\lambda}\tilde{y}_k^2\right)\Psi_T$$

Using the way, Ref. [31], of calculation of this equation integrals, we obtain:

$$\int\prod_{k=1}^{N}dx_kdy_k(x_k-x_0)\Psi_T^*\frac{\partial\Psi_T}{\partial\tilde{t}}=$$
$$N\left(\frac{\dot{x}_0}{2}+iA_1q_1^2\dot{x}_0+i\frac{q_1^2}{2}\dot{C}_1\right)$$

and

$$\int\prod_{k=1}^{N}dx_kdy_k(y_k-y_0)\Psi_T^*\frac{\partial\Psi_T}{\partial\tilde{t}}=$$
$$N\left(\frac{\dot{y}_0}{2}+iA_2q_2^2\dot{y}_0+i\frac{q_2^2}{2}\dot{C}_2\right)$$

Then

$$\int\prod_{k=1}^{N}dx_kdy_k(x_k-x_0)\Psi_T^*\sum_{k=1}^{N}\tilde{\Delta}_k\Psi_T=$$
$$-N(iC_1-2C_1A_1q_1^2)$$
$$\int\prod_{k=1}^{N}dx_kdy_k(y_k-y_0)\Psi_T^*\sum_{k=1}^{N}\tilde{\Delta}_k\Psi_T=$$
$$-N(iC_2-2C_2A_2q_2^2),$$

and

$$\int \prod_{k=1}^{N} dx_k dy_k (x_k - x_0)\Psi_T^* \sum_{k=1}^{N} \tilde{x}_k^2 \Psi_T = N x_0 q_1^2$$

$$\int \prod_{k=1}^{N} dx_k dy_k (y_k - y_0)\Psi_T^* \sum_{k=1}^{N} \tilde{y}_k^2 \Psi_T = N y_0 q_2^2$$

5. AVERAGE QUANTITIES FOR $f_{x,y}^1$ WITH ANYON PART OF HAMILTONIAN $\hat{H}_1$.

We start with the expression for the square of modulo of wave function $\Psi_T(\vec{R},t)$, Eq. (12), (for the simplicity, everywhere below, we omit signs tilte). It equals to

$$\left|\Psi_T(\vec{R})\right|^2 = \left(\frac{1}{\pi q_1 q_2}\right)^N \prod_{k=1}^{N} \exp\left[-\frac{x_{k0}^2}{q_1^2} - \frac{y_{k0}^2}{q_2^2}\right]$$

where $x_{k0} = x_k - x_0$ and $y_{k0} = y_k - y_0$.

First, we need to calculate the integral in the average quantities for $f_{x,y}^1 = y_{k0}$, related to term

$$y_{k0}\Psi_T^* \frac{\vec{r}_{kj}\cdot\vec{\nabla}_k}{\left|\vec{r}_{kj}\right|^2}\Psi_T = \frac{y_{k0}\left|\Psi_T(\vec{R})\right|^2}{x_{kj}^2 + y_{kj}^2} Z_1,$$

where

$$Z_1 = -2x_{kj}x_{k0}\left(\frac{1}{2q_1^2} + iA_1\right) + iC_1 x_{kj} - 2y_{kj}y_{k0}\left(\frac{1}{2q_2^2} + iA_2\right) + iC_2 y_{kj}$$

The expression for this integral is:

$$I_{Z_1}^y = \left(\frac{1}{\pi q_1 q_2}\right)^2 \sum_{k=1}^{N} \sum_{j\neq k} \iiint \int dx_k dy_k dx_j dy_j \times \frac{y_{k0}}{x_{kj}^2 + y_{kj}^2} Z_1 \exp\left[-\frac{x_{k0}^2}{q_1^2} - \frac{y_{k0}^2}{q_2^2} - \frac{x_{j0}^2}{q_1^2} - \frac{y_{j0}^2}{q_2^2}\right]$$

We introduce new variables $x_{kj} = x_k - x_j$ and $y_{kj} = y_k - y_j$ then $dx_j = -dx_{kj}$ and $dy_j = -dy_{kj}$ and taking into account that

$$\exp\left[-\frac{x_{j0}^2}{q_1^2} - \frac{y_{j0}^2}{q_2^2}\right]$$

$$\frac{y_{kj}^2 - 2y_{kj}y_{k0} + y_{k0}^2}{q_2^2}\Bigg]$$

the exponential function in the Eq. (24) will have the form:

$$\exp\left[-\frac{2x_{k0}^2}{q_1^2}-\frac{2y_{k0}^2}{q_2^2}-\frac{x_{kj}^2-2x_{kj}x_{k0}}{q_1^2}-\frac{y_{kj}^2-2y_{kj}y_{k0}}{q_2^2}\right]$$

Next, substituting expressions for Z_1, Eq. (23), and exponential function, Eq. (26), in Eq. (24) for integral $I_{Z_1}^y$ then, using the formula, Eq. (44), at the integration of $I_{Z_1}^y$ over $dx_k dy_k dx_{kj} dy_{kj}$, we find that only the last term $iC_2 y_{kj}$ of Z_1 gives a non zero contribution into $I_{Z_1}^y$. And its expression is:

$$I_{Z_1}^y = iN(N-1)C_2.$$

At the derivation of this expression for $I_{Z_1}^y$, we have used the formulas:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx_{kj}dy_{kj}\frac{x_{kj}^2}{x_{kj}^2+y_{kj}^2}e^{-\frac{\alpha_1}{2}x_{kj}^2-\frac{\beta_1}{2}y_{kj}^2} = \frac{2\pi}{\sqrt{\alpha_1\beta_1}}+\pi\sqrt{\frac{\beta_1}{\alpha_1}}\frac{1}{(\alpha_1/2-\beta_1/2)}$$

for $\alpha_1 > \beta_1$ and

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx_{kj}dy_{kj}\frac{x_{kj}^2}{x_{kj}^2+y_{kj}^2}e^{-\frac{\alpha_1}{2}x_{kj}^2-\frac{\beta_1}{2}y_{kj}^2} = \frac{2\pi}{\sqrt{\alpha_1\beta_1}}+\pi\sqrt{\frac{\alpha_1}{\beta_1}}\frac{1}{(\beta_1/2-\alpha_1/2)}$$

for $\beta_1 > \alpha_1$, which can be obtained, using expressions Eqs. (30) - (32):

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx_{kj}dy_{kj}\frac{x_{kj}^2}{x_{kj}^2+y_{kj}^2}e^{-\frac{\alpha_1}{2}x_{kj}^2-\frac{\beta_1}{2}y_{kj}^2}$$

At the calculation of that integral over dx_{kj}, we use the formula 3.466.2 of the book [32]:

$$\int_0^{+\infty} dx\frac{x^2}{x^2+\beta^2}e^{-\mu^2x^2} = \frac{\pi}{2\mu}-\frac{\pi\beta}{2}e^{\mu^2\beta^2}[1-\mathrm{erf}(\beta\mu)]$$

with $[\mathrm{Re}\,\beta > 0, |\arg\mu < \pi/4|]$ and $\mathrm{erf}(x)$ is the error function.

At the calculation of obtained integral over the dy_{kj}, we use the formula 6.289.2 of the same book [32] of I.S. Gradshteyn and I.M. Ryzhik:

$$\int_0^{+\infty}\mathrm{erf}(\beta x)e^{(\beta^2-\mu^2)x^2}x dx = \frac{\beta}{2\mu(\mu^2-\beta^2)},$$

at $[\mathrm{Re}\,\mu^2 > \mathrm{Re}\,\beta^2, |\arg\mu < \pi/4|]$

In the analogous way, one may calculate the expression for integral $I_{Z_1}^x$ for the average quantities of $f_{x,y}^1 = x_{k0}$. It equals to expression:

$$I_{Z_1}^x = iN(N-1)C_1.$$

6. AVERAGE QUANTITIES FOR $f^1_{x,y}$ WITH HAMILTONIAN $\hat{\tilde{H}}_2$.

We calculate the integral in the average quantities for $f^1_{x,y} = y_{k0}$, related to term

$$y_{k0}\Psi_T^*\vec{A}_\nu(\vec{r}_k)\cdot\vec{\nabla}_k\Psi_T = \nu y_{k0}\frac{|\Psi_T|^2}{x_{kj}^2+y_{kj}^2}Z_2,$$

where

$$Z_2 = 2y_{kj}x_{k0}\left(\frac{1}{2q_1^2}+iA_1\right) - iC_1y_{kj} - 2x_{kj}y_{k0}\left(\frac{1}{2q_2^2}+iA_2\right)+iC_2x_{kj}$$

Again, the expression for this integral is:

$$I^y_{Z_2} = \nu\left(\frac{1}{\pi q_1q_2}\right)^2\sum_{k=1}^{N}\sum_{j\neq k}\iiint\int dx_kdy_kdx_jdy_j \times \frac{y_{k0}}{x_{kj}^2+y_{kj}^2}Z_2exp\left[-\frac{x_{k0}^2}{q_1^2}-\frac{y_{k0}^2}{q_2^2}-\frac{x_{j0}^2}{q_1^2}-\frac{y_{j0}^2}{q_2^2}\right]$$

We follow the procedure of calculation, discribed from Eq. (24) up to Eq. (29), except of substituting expressions for Z_2, Eq. (35), and exponential function, Eq. (26), in Eq. (36) for integral $I^y_{Z_2}$ then, using the formula, Eq. (44), at the integration of $I^y_{Z_2}$ over $dx_kdy_kdx_{kj}dy_{kj}$, we find that only the term $-iC_1y_{kj}$ of Z_2 gives a non zero contribution into $I^y_{Z_2}$. We obtain

$$I^y_{Z_2} = -i\nu N(N-1)C_1.$$

Analogously, we calculate the expression for integral $I^x_{Z_2}$ for the average quantities of $f^1_{x,y} = x_{k0}$. It equals to expression:

$$I^x_{Z_2} = i\nu N(N-1)C_2.$$

We calculate the integral in the average quantities for $f^1_{x,y} = y_{k0}$, related to the last term in the Hamiltonian $\hat{H}_2$. It is:

$$\sum_{k=1}^{N}\nu\sum_{j\neq k}\frac{\overrightarrow{A_\nu}(\overrightarrow{r_k})r_{kj}}{|r_{kj}|^2}y_{k0}|\Psi_T|^2 = \nu^2\sum_{k=1}^{N}\sum_{j\neq k}\sum_{l\neq k}\frac{-y_{kl}x_{kj}+x_{kl}y_{kj}}{(x_{kj}^2+y_{kj}^2)(x_{kl}^2+y_{kl}^2)}y_{k0}Z_{exp},$$

where

$$Z_{exp} = \exp\left[-\frac{x_{k0}^2}{q_1^2}-\frac{y_{k0}^2}{q_2^2}-\frac{x_{j0}^2}{q_1^2}-\frac{y_{j0}^2}{q_2^2}\right]\times \exp\left[-\frac{x_{l0}^2}{q_1^2}-\frac{y_{l0}^2}{q_2^2}\right].$$

Introducing variables x_{kj} and y_{kj}, we expressed the first exponential function in Eq. (40) in the form of Eq. (26). Now, we introduce variables x_{kl} and y_{kl} then

$$\exp\left[-\frac{2x_{k0}^2}{q_1^2}-\frac{2y_{k0}^2}{q_2^2}-\frac{x_{l0}^2}{q_1^2}-\frac{y_{l0}^2}{q_2^2}\right]$$

will have a form:

$$\exp\left[-\frac{3x_{k0}^2}{q_1^2}-\frac{3y_{k0}^2}{q_2^2}-\frac{x_{kl}^2-2x_{kl}x_{k0}}{q_1^2}-\frac{y_{kl}^2-2y_{kl}y_{k0}}{q_2^2}\right].$$

Introducing in Eq. (41) last two parts inside of exponential function, Eq. (26), we find the final expression of the function Z_{exp}

$$\exp\left[-\frac{3x_{k0}^2}{q_1^2}-\frac{3y_{k0}^2}{q_2^2}-\frac{x_{kj}^2+x_{kl}^2-2(x_{kj}+x_{kl})x_{k0}}{q_1^2}-\frac{y_{kj}^2+y_{kl}^2-2(y_{kj}+y_{kl})y_{k0}}{q_2^2}\right]$$

Our goal is to calculate the integral:

$$I_{Z_{exp}}^{y}=\iiint\iiint dx_k dy_k dx_{kj} dy_{kj} dx_{kl} dy_{kl}\frac{-y_{kl}x_{kj}+x_{kl}y_{kj}}{(x_{kj}^2+y_{kj}^2)(x_{kl}^2+y_{kl}^2)}y_{k0}Z_{exp}$$

Using a formula below:

$$\int_{-\infty}^{+\infty}x^n e^{-px^2+2qx}dx=n!\,e^{\frac{q^2}{p}}\sqrt{\frac{\pi}{p}}\left(\frac{q}{p}\right)^n\sum_{k=0}^{E\left(\frac{n}{2}\right)}\frac{1}{(n-2k)!\,(k)!}\left(\frac{p}{4q^2}\right)^k,$$

we find

$$\int_{-\infty}^{+\infty}dx_{k0}\exp\left[-\frac{3x_{k0}^2}{q_1^2}+\frac{2(x_{kj}+x_{kl})x_{k0}}{q_1^2}\right]=q_1\sqrt{\frac{\pi}{3}}\exp\left[\frac{(x_{kj}+x_{kl})^2}{3q_1^2}\right]$$

and

$$\int_{-\infty}^{+\infty}dy_{k0}y_{k0}\exp\left[-\frac{3y_{k0}^2}{q_2^2}+\frac{2(y_{kj}+y_{kl})y_{k0}}{q_2^2}\right]=q_2\sqrt{\frac{\pi}{3}}\exp\left[\frac{(y_{kj}+y_{kl})^2}{3q_2^2}\right]\frac{(y_{kj}+y_{kl})}{3}$$

Taking into account the expressions Eq. (45) and Eq. (46), the integral $I_{Z_{exp}}^{y}$ transforms into form:

$$I^{y}_{Z_{exp}} = \frac{\pi}{3} q_1 q_2 \iiint \int dx_{kj} dy_{kj} dx_{kl} dy_{kl} \times \frac{-y_{kl}x_{kj} + x_{kl}y_{kj}}{\left(x_{kj}^2 + y_{kj}^2\right)\left(x_{kl}^2 + y_{kl}^2\right)} \frac{\left(y_{kj} + y_{kl}\right)}{3} Z_{exp}$$

with the new expression for Z_{exp} :

$$Z_{\text{exp}} = \exp\left[-\frac{2}{3q_1^2}\left(x_{kj}^2 + x_{kl}^2 - x_{kj}x_{kl}\right) - \frac{2}{3q_2^2}\left(y_{kj}^2 + y_{kl}^2 - y_{kj}y_{kl}\right)\right].$$

The first term in the sum $-y_{kl}x_{kj} + x_{kl}y_{kj}$ of the last expression for $I^{y}_{Z_{exp}}$ does not depend on variable x_{kl}. Therefore, we can calculate the integral

$$I_{x_{kl}} = \int_{-\infty}^{+\infty} dx_{kl} \frac{1}{x_{kl}^2 + y_{kl}^2} e^{-\frac{2\alpha_1}{3}\left(x_{kl}^2 - x_{kj}x_{kl}\right)}.$$

For this purpose, we use the definition of Gamma function $\Gamma(x)$:

$$\frac{\Gamma(x)}{a^x} = \int_0^{+\infty} d\tau \tau^{x-1} e^{-a\tau}$$

to rewrite the $I_{x_{kl}}$ in the form:

$$I_{x_{kl}} = \frac{1}{\Gamma(1)} \int_0^{+\infty} d\tau e^{-y_{kl}^2\tau} \int_{-\infty}^{+\infty} dx_{kl} \times \exp\left[-\left(\frac{2\alpha_1}{3} + \tau\right) x_{kl}^2 + \frac{2\alpha_1}{3} x_{kj}x_{kl}\right]$$

Using again a formula, Eq. (44), one obtains the result for $I_{x_{kl}}$:

$$I_{x_{kl}} = \frac{\sqrt{\pi}}{\Gamma(1)} \int_0^{+\infty} d\tau \frac{e^{-y_{kl}^2\tau}}{\sqrt{2\alpha_1/3 + \tau}} \times \exp\left[\left(\frac{\alpha_1 x_{kj}}{3}\right)^2 / (2\alpha_1/3 + \tau)\right]$$

In this expression for $I_{x_{kl}}$, at variation of variable τ in the limits from 0 up to $+\infty$, the function

$$\exp\left[\left(\frac{\alpha_1 x_{kj}}{3}\right)^2 / (2\alpha_1/3 + \tau)\right]$$

will change from $e^{\alpha_1 x_{kj}^2/3}$up$_2$ to 1 . However, at $\tau \to +\infty$ limit, the function $e^{-y_{kl}^2\tau}$ will make zero a whole integrand of $I_{x_{kl}}$. Therefore, one can assume

$$\exp\left[\left(\frac{\alpha_1 x_{kj}}{3}\right)^2 / (2\alpha_1/3 + \tau)\right] \approx e^{\alpha_1 x_{kj}^2/3}$$

and thus take the approximate expression for integral

$$I_{x_{kl}} \approx \frac{\sqrt{\pi}}{\Gamma(1)} e^{\alpha_1 x_{kj}^2/3} \int_0^{+\infty} d\tau \frac{e^{-y_{kl}^2\tau}}{\sqrt{2\alpha_1/3 + \tau}}$$

We take into account the exponential function $e^{\alpha_1 x_{kj}^2/3}$ from Eq. (52) in the factor $e^{-2\alpha_1/3x_{kj}^2}$, where $\alpha_1 = 1/q_1^2$, of the expression Z_{exp}, and together with obtained this factor the integral over dx_{kj} of $I^y_{Z_{exp}}$ with the first term in the sum $-y_{kl}x_{kj} + x_{kl}y_{kj}$ gives:

$$\int_{-\infty}^{+\infty} dx_{kj} \frac{x_{kj}}{x_{kj}^2 + y_{kj}^2} e^{-\frac{\alpha_1}{3}x_{kj}^2} = 0$$

and therefore, we find that $I^y_{Z_{exp}} = 0$.

We consider the second term of the sum $-y_{kl}x_{kj} + x_{kl}y_{kj}$ in the integrand of $I^y_{Z_{exp}}$. It is easy to show that

$$\frac{d}{d(2\alpha_1 x_{kj}/3)} I_{x_{kl}} = \int_{-\infty}^{+\infty} dx_{kl} \frac{x_{kl}}{x_{kl}^2 + y_{kl}^2} e^{-\frac{2\alpha_1}{3}(x_{kl}^2 - x_{kj}x_{kl})},$$

if expression for $I_{x_{kl}}$ is taken from Eq. (49). However, from the final expression for $I_{x_{kl}}$, Eq. (52), one obtains $d/d(2\alpha_1 x_{kj}/3) I_{x_{kl}} \sim x_{kj} e^{\alpha_1 x_{kj}^2/3}$. Therefore, using Eq. (53), we find again $I^y_{Z_{exp}} = 0$.

We demonstrated $I^y_{Z_{exp}} = 0$ at calculating the average quantities for $f^1_{x,y} = y_{k0}$, related to the last term in the Hamiltonian $\hat{H}_2$. One can show that the same average quantities, however, calculating now for $f^1_{x,y} = x_{k0}$, give also $I^x_{Z_{exp}} = 0$. To get this result we used the expression

$$I^x_{Z_{exp}} = \frac{\pi}{3} q_1 q_2 \iiint \int dx_{kj} dy_{kj} dx_{kl} dy_{kl} \times \frac{-y_{kl}x_{kj} + x_{kl}y_{kj}}{(x_{kj}^2 + y_{kj}^2)(x_{kl}^2 + y_{kl}^2)} \frac{(x_{kj} + x_{kl})}{3} Z_{exp}$$

and

$$\int_{-\infty}^{+\infty} dy_{kj} \frac{y_{kj}}{x_{kj}^2 + y_{kj}^2} e^{-\frac{\alpha_1}{3}y_{kj}^2} = 0$$

7. HARMONIC OSCILLATOR EQUATION FOR THE CENTRE-OF-MASS.

Substituting in the Schrödinger equation, Eq. (13), results of average quantities for $f^1_{x,y}$, calculated in the above three sections, we find the equations of motion for the x_0 coordinate

$$i\frac{\dot{x}_0}{2} - A_1 q_1^2 \dot{x}_0 - \frac{q_1^2}{2}\dot{C}_1 = i\frac{C_1}{2} - C_1 A_1 q_1^2 + \frac{\lambda_1}{2} x_0 q_1^2 - i\nu(N-1)C_1 + \nu(N-1)C_2$$

and y_0 coordinate

$$i\frac{\dot{y}_0}{2} - A_2 q_2^2 \dot{y}_0 - \frac{q_2^2}{2}\dot{C}_2 = i\frac{C_2}{2} - C_2 A_2 q_2^2 + \frac{\lambda_2}{2} y_0 q_2^2 - i\nu(N-1)C_2 - \nu(N-1)C_1$$

of the centre-of-mass. In equations Eqs. (57) - (58) $\lambda_1 = \lambda$ and $\lambda_2 = 1/\lambda$

Equating imaginary parts of both these equations, we find:

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$$\frac{\dot{x}_0}{2} = \frac{C_1}{2} - \nu(N-1)C_1$$
$$\frac{\dot{y}_0}{2} = \frac{C_2}{2} - \nu(N-1)C_2,$$

from where

$$C_1 = \frac{\dot{x}_0}{b}$$
$$C_2 = \frac{\dot{y}_0}{b}$$

Here, we introduced the constant $b = 1 - 2\nu(N-1)$. From Eq. (60), we express $\dot{x}_0$ and $\dot{y}_0$ through the C_1 and C_2, respectively, and substitute them in the real parts of Eq. (57) and Eq. (58). We obtain

$$\dot{C}_1 + \lambda_1 x_0 = 2\nu(N-1)\left(2C_1A_1 - \frac{C_2}{q_1^2}\right)$$
$$\dot{C}_2 + \lambda_2 y_0 = 2\nu(N-1)\left(2C_2A_2 + \frac{C_1}{q_2^2}\right).$$

Our goal is to consider solution of these equations on the first order small quantities, therefore, we omit the C_1A_1 and C_2A_2 terms from the consideration and assume that $q_1^2 = q_{10}^2$ and $q_2^2 = q_{20}^2$. Taking into account the relationship, Eq. (60), we write a set of equations:

$$\frac{\ddot{x}_0}{b} + \lambda_1 x_0 = -\frac{2\nu(N-1)\dot{y}_0}{q_{10}^2 b}$$
$$\frac{\ddot{y}_0}{b} + \lambda_2 y_0 = \frac{2\nu(N-1)\dot{x}_0}{q_{20}^2 b}.$$

We try to find the solutions in the form $x_0 \sim e^{i\omega t}$ and $y_0 \sim e^{i\omega t}$ then Eqs. (62) reduce to

$$-\tilde{\omega}^2 + \tilde{\lambda_1} = -iK_1\tilde{\omega}$$
$$-\tilde{\omega}^2 + \tilde{\lambda_2} = iK_2\tilde{\omega},$$

where $\tilde{\omega} = \omega/b, \tilde{\lambda_1} = \lambda_1/b, \tilde{\lambda_2} = \lambda_2/b, K_1 = 2\nu(N-1)\,/(bq_{10}^2)$ and $K_2 = 2\nu(N-1)/(bq_{20}^2)$.
Multiplying two equations of Eq. (63) to each other, one obtains

$$\left(-\tilde{\omega}^2 + \tilde{\lambda_1}\right)\left(-\tilde{\omega}^2 + \tilde{\lambda_2}\right) = K_1K_2\tilde{\omega}^2$$

and thus the equation

$$\tilde{\omega}^4 - \tilde{\omega}^2\left(\tilde{\lambda_1} + \tilde{\lambda_2} + K_1K_2\right) + \tilde{\lambda_1}\tilde{\lambda_2} = 0.$$

The solution of this equation is:

$$(\tilde{\omega}^2)_{1,2} = \frac{1}{2}\left(\tilde{\lambda_1} + \tilde{\lambda_2} + K_1K_2\right) \pm \left[\frac{1}{4}\left(\tilde{\lambda_1} + \tilde{\lambda_2} + K_1K_2\right)^2 - \tilde{\lambda_1}\tilde{\lambda_2}\right].$$

We analyse an effect of the different cases of statistics of particles ν and an harmonic potential anysotropy λ_1 and λ_2 on centre-of-mass oscillatory frequency ω^2. Let assume that we consider the system of bosons $\nu = 0$. For this case of particle statistics, $b = 1, K_1 = K_2 = 0$ and from equation

$$(\tilde{\omega}^2)_{1,2} = \omega_{1,2}^2 = 1/2(\lambda_1 + \lambda_2) \pm 1/2(\lambda_1 - \lambda_2),$$

we have

$$\omega_1^2 = \lambda_1$$
$$\omega_2^2 = \lambda_2.$$

For the case $\nu = 0$ and isotropic harmonic potential $\lambda_1 = \lambda_2 = 1$, we have $\omega_1^2 = \omega_2^2 = 1$.

For the system of anyons $\nu \neq 0$ and arbitrary harmonic potential λ_1 and λ_2, the centre-of-mass oscillatory frequences are determined by Eq. (64).

8. CONCLUSION

So, after scrutinizing problems and tasks this work and with the help of acquired results the following statements can be done to conclude the work:

- In order to perform the tasks set up in this research a new method for calculating problems has been utilized. It is a cumulant method to get the width equation and equation of the motion for the centre of mass to get the collective frequencies. This method gave an opportunity to avoid for writing an action and take a variation over it and also solve a differential equation of the second order which is an overwhelming task. Instead, we solved integrals of Gaussian type which is much easier than a variational approach.
- By utilizing the cumulant method, the expression for the centre-of-mass oscillatory frequencies for the system of anyons and arbitrary harmonic potential has been derived in the last chapter. Furthemore, special cases for the determined frequency have been considered with the parameters ν and b, K_1, K_2.

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